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Efficient loop-check for multimodal $KD45_n$ logic

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Abstract. We introduce sequent calculus for multi-modal logic $KD45_n$ which uses efficient loop-check. Efficiency of the used loop-check is obtained by using marked modal operator $\Box_i^\bullet$ which is used as an alternative to sequent with histories ([2,3]). We use inference rules with *or* branches to make all rules invertible or semi-invertible. We show the maximum height of the constructed derivation tree. Also polynomial space complexity is proved.

Keywords: sequent calculus, multi-modal logic $KD45_n$, efficient loop-check.

1. Introduction

Multi-modal logic $KD45_n$ is a part of the widely used BDI logic, described in [5]. There is known sequent calculus for logic $KD45_n$, but it uses inefficient (direct) loop-check. Direct loop-check technique used in sequent calculus requires to check all the sequents in the current branch after every rule application. It means, that most of the time is used to compare sequents instead of applying inference rules. The main goal of the efficient loop-check is to make it work 'locally'. This can be achieved by using some properties of the used logic. Basic principle of 'local' loop-check is to store some more information in the sequent, and later, use this information for loop test. Sequents with histories can be used for this. Such an approach is used in [2,3], where efficient loop-check for some modal logics is shown. In our work, we use marked modal operator $\Box_i^\bullet$ as an alternative to histories.

Sequent calculus for multi-modal logic $KD45_n$ uses n modal operators $\Box_1, \Box_2, \ldots, \Box_n$ and has the following rules:

$$\frac{\phi, \Gamma \to \Delta \quad \psi, \Gamma \to \Delta}{\phi \vee \psi, \Gamma \to \Delta}\ (\vee L) \quad \frac{\phi, \psi, \Gamma \to \Delta}{\phi \& \psi, \Gamma \to \Delta}\ (\& L) \quad \frac{\Gamma \to \phi, \Delta}{\neg\phi, \Gamma \to \Delta}\ (\neg L)$$

$$\frac{\Gamma \to \phi, \Delta \quad \Gamma \to \psi, \Delta}{\Gamma \to \phi \& \psi, \Delta}\ (\& R) \quad \frac{\Gamma \to \phi, \psi, \Delta}{\Gamma \to \phi \vee \psi, \Delta}\ (\vee R) \quad \frac{\phi, \Gamma \to \Delta}{\Gamma \to \neg\phi, \Delta}\ (\neg R)$$

$$\frac{\Gamma \to \Delta}{\Gamma, \Gamma' \to \Delta, \Delta'}\ (Weak) \qquad \frac{\Gamma, \Box_i\Gamma \to \Theta, \Box_i\Theta, \Box_i\Delta}{\Box_i\Gamma \to \Box_i\Theta, \Box_i\Delta}\ (\Box_i)$$

(Θ is empty or only one formula)

DEFINITION 1. Sequent calculus with an axiom $\Gamma, \phi \to \Delta, \phi$ and rules $(\vee L)$, $(\vee R)$, $(\& L)$, $(\& R)$, $(\neg L)$, $(\neg R)$, $(Weak)$, $(\Box_i)$ we call $KD45_n$.

Table 1

Calculus	Used rules
$KD45_n$	$(Weak) + (\Box_i)$
$\alpha\beta - KD45_n$	$(\Box or) + (\alpha\Box_i) + (\beta\Box_i)$
$\alpha\beta\gamma\delta - KD45_n^\bullet$	$(\Box^\bullet or) + (\alpha\Box_i^\bullet) + (\beta\Box_i^\bullet) + (\gamma\Box_i^\bullet) + (\delta\Box_i^\bullet)$
$\alpha\beta - KD45_n^\bullet$	$(\Box^\bullet or) + (\alpha\Box_i^\bullet) + (\beta\Box_i^\bullet)$

According to procedures described in [4], we can use sequent calculus $KD45_n$ with backward proof search and loop-check to get sequent derivability. Unfortunately, used loop-check takes most of the calculation time, because after every inference rule application we have to check all the sequents in the current branch. In the next section, we propose a new sequent calculus which uses very restricted loop-check.

2. Sequent calculus $\alpha\beta - KD45_n^\bullet$

First of all, we introduce sequent calculus which uses only invertible or semi-invertible rules. For this reason we use rules with *or* branches. Such a rule contains several premises separated by || (conclusion is derivable if at least one of the premises is derivable). After we introduce marked operator $\Box_i^\bullet$, which will be used together with operator $\Box_i$. Marked operator $\Box_i^\bullet$ is used to store information about modal rules applications. New operator $\Box_i^\bullet$ let us to introduce sequent calculus which uses 4 different modal rules instead of the $(\Box_i)$. Finally we show that two modal rules (the only those can be cause of the loops) are redundant and can be removed from the calculus.

Differences in used rules for defined calculus are shown in the Table 1.

DEFINITION 2. Sequent calculus with an axiom $\Gamma, \phi \to \Delta, \phi$ and rules $(\vee L)$, $(\vee R)$, $(\& L)$, $(\& R)$, $(\neg L)$, $(\neg R)$, $(\Box or)$, $(\alpha\Box_i)$, $(\beta\Box_i)$ we call $\alpha\beta - KD45_n$.

$$\frac{\Lambda \to \Pi \quad || \quad \Box_1\Gamma_1 \to \Box_1\Delta_1 \quad || \dots || \quad \Box_n\Gamma_n \to \Box_n\Delta_n}{\Lambda, \Box_1\Gamma_1, \dots, \Box_n\Gamma_n \to \Pi, \Box_1\Delta_1, \dots, \Box_n\Delta_n} \quad (\Box or)$$

(Λ, Π - contains only non modalized atomic formulas)

$$\frac{\Gamma, \Box_i\Gamma \to \phi_1, \Box_i\Delta \quad || \dots || \quad \Gamma, \Box_i\Gamma \to \phi_m, \Box_i\Delta}{\Box_i\Gamma \to \Box_i\Delta} \quad (\alpha\Box_i) \qquad \frac{\Gamma, \Box_i\Gamma \to}{\Box_i\Gamma \to} \quad (\beta\Box_i)$$

$(\Box_i\Delta = \Box_i\phi_1, \dots, \Box_i\phi_m, \quad m > 0)$

LEMMA 1. *A sequent S is derivable in sequent calculus $KD45_n$ if and only if S is derivable in sequent calculus $\alpha\beta - KD45_n$.*

Proof. In sequent calculus $KD45_n$ we can use rule $(Weak)$ only together with rule $(\Box_i)$ application without loosing derivability. In such a case, rule $(Weak)$ can be replaced by rule $(\Box or)$ if we use sequent derivation tree with *or* branches. Rules $(\alpha\Box_i)$ and $(\beta\Box_i)$ are just two possible cases of the rule $(\Box_i)$ if we use sequent derivation tree with *or* branches.

DEFINITION 3. Sequent calculus with an axiom $\Gamma, \phi \to \Delta, \phi$ and rules $(\vee L)$, $(\vee R)$, $(\& L)$, $(\& R)$, $(\neg L)$, $(\neg R)$, $(\Box^\bullet or)$, $(\alpha\Box_i^\bullet)$, $(\beta\Box_i^\bullet)$, $(\gamma\Box_i^\bullet)$, $(\delta\Box_i^\bullet)$ we call $\alpha\beta\gamma\delta - KD45_n^\bullet$.

$$\frac{\Lambda \to \Pi \quad || \quad \Box_1\Gamma_1, \Box_1^\bullet\Gamma_1' \to \Box_1\Delta_1, \Box_1^\bullet\Delta_1' \quad || \dots || \quad \Box_n\Gamma_n, \Box_n^\bullet\Gamma_n' \to \Box_n\Delta_n, \Box_n^\bullet\Delta_n'}{\Lambda, \Box_1\Gamma_1, \Box_1^\bullet\Gamma_1', \dots, \Box_n\Gamma_n, \Box_n^\bullet\Gamma_n' \to \Pi, \Box_1\Delta_1, \Box_1^\bullet\Delta_1', \dots, \Box_n\Delta_n, \Box_n^\bullet\Delta_n'} \ (\Box^\bullet or)$$

(Λ, Π - contains only non modalized atomic formulas)

$$\frac{\Gamma, \Gamma', \Box_i^\bullet\Gamma, \Box_i^\bullet\Gamma' \to \phi_1, \Box_i^\bullet\Delta, \Box_i^\bullet\Delta' \quad || \dots || \quad \Gamma, \Gamma', \Box_i^\bullet\Gamma, \Box_i^\bullet\Gamma' \to \phi_k, \Box_i^\bullet\Delta, \Box_i^\bullet\Delta'}{\Box_i\Gamma, \Box_i^\bullet\Gamma' \to \Box_i\Delta, \Box_i^\bullet\Delta'} \ (\alpha\Box_i^\bullet)$$

($\Box_i\Delta = \Box_i\phi_1, \dots, \Box_i\phi_m, \quad \Box_i^\bullet\Delta = \Box_i^\bullet\phi_1, \dots, \Box_i^\bullet\phi_m, \quad \Box_i^\bullet\Delta' = \Box_i^\bullet\phi_{m+1}, \dots, \Box_i^\bullet\phi_k, \quad m > 0$)

(rule $(\alpha\Box_i^\bullet)$ can be applied only if $\Box_i\Gamma \cup \Box_i\Delta \neq \emptyset$)

$$\frac{\Gamma, \Gamma', \Box_i^\bullet\Gamma, \Box_i^\bullet\Gamma' \to}{\Box_i\Gamma, \Box_i^\bullet\Gamma' \to} \ (\beta\Box_i^\bullet)$$

(rule $(\beta\Box_i^\bullet)$ can be applied only if $\Box_i\Gamma \neq \emptyset$)

$$\frac{\Gamma', \Box_i^\bullet\Gamma' \to \phi_1, \Box_i^\bullet\Delta' \quad || \dots || \quad \Gamma', \Box_i^\bullet\Gamma' \to \phi_k, \Box_i^\bullet\Delta'}{\Box_i^\bullet\Gamma' \to \Box_i^\bullet\Delta'} \ (\gamma\Box_i^\bullet) \qquad \frac{\Gamma', \Box_i^\bullet\Gamma' \to}{\Box_i^\bullet\Gamma' \to} \ (\delta\Box_i^\bullet)$$

($\Box_i^\bullet\Delta' = \Box_i^\bullet\phi_1, \dots, \Box_i^\bullet\phi_k$)

LEMMA 2. *A sequent S is derivable in sequent calculus $\alpha\beta - KD45_n$ if and only if S is derivable in sequent calculus $\alpha\beta\gamma\delta - KD45_n^\bullet$.*

Proof. Rule $(\Box^\bullet or)$ is the same rule $(\Box or)$, it just uses marked operator $\Box_i^\bullet$. Rules $(\alpha\Box_i^\bullet)$ and $(\gamma\Box_i^\bullet)$ are just two possible cases of the rule $(\alpha\Box_i)$. Rules $(\beta\Box_i^\bullet)$ and $(\delta\Box_i^\bullet)$ are just two possible cases of the rule $(\beta\Box_i)$.

DEFINITION 4. Sequent calculus with an axiom $\Gamma, \phi \to \Delta, \phi$ and rules $(\vee L)$, $(\vee R)$, $(\& L)$, $(\& R)$, $(\neg L)$, $(\neg R)$, $(\Box^\bullet or)$, $(\alpha\Box_i^\bullet)$, $(\beta\Box_i^\bullet)$ we call $\alpha\beta - KD45_n^\bullet$.

LEMMA 3. *A sequent S is derivable in sequent calculus $\alpha\beta\gamma\delta - KD45_n^\bullet$ if and only if S is derivable in sequent calculus $\alpha\beta - KD45_n^\bullet$.*

Proof. If we get sequent $S' = \Box_i^\bullet\Gamma' \to \Box_i^\bullet\Delta'$ somewhere in the derivation tree for sequent calculus $\alpha\beta\gamma\delta - KD45_n^\bullet$, we can treat it as not derivable and do not proceed with this branch. Even if sequent S' is derivable in $\alpha\beta\gamma\delta - KD45_n^\bullet$, according to [1], we can always find another *or* branch bellow S' which is derivable. So, rules $(\gamma\Box_i^\bullet)$ and $(\delta\Box_i^\bullet)$ are redundant.

Derivation tree construction in sequent calculus $\alpha\beta - KD45_n^\bullet$ always terminates, because after every rule $(\alpha\Box_i^\bullet)$ (or $(\beta\Box_i^\bullet)$) application we get one more formula modalized with marked operator $\Box_i^\bullet$. Marked operator $\Box_i^\bullet$ cannot become unmarked. So, finally we get only marked modalized formulas in the sequent, but for such a sequent rules $(\alpha\Box_i^\bullet)$, $(\beta\Box_i^\bullet)$ cannot be applied.

Marked operator $\Box_i^\bullet$ is a kind of history, because it stores information about rules applied bellow. If we get sequent which is not an axiom, and which contains only marked modalized formulas, we know that some loop exists on that branch.

3. Complexity

DEFINITION 5. We define length function len as follows:
$len(\phi) = 1$, if ϕ is propositional variable,
$len(\neg\phi) = len(\Box_i\phi) = len(\Box_i^\bullet\phi) = len(\phi) + 1$,
$len(\phi \vee \psi) = len(\phi \& \psi) = len(\phi) + len(\psi) + 1$.
Length of the sequent is sum of formulas lengths.

LEMMA 4. *If sequent S contains k logical operators and m modal operators $\Box_i$, then the maximum height of the derivation tree in $\alpha\beta - KD45_n^\bullet$ is $m \cdot k$.*

Proof. Any sequent in a derivation tree can contain only subformulas of the initial sequent S. Every rule $(\alpha\Box_i^\bullet)$ or $(\beta\Box_i^\bullet)$ changes at least one subformula of the shape $\Box_i\phi$ into $\Box_i^\bullet\phi$. So, at most m times rules $(\alpha\Box_i^\bullet)$, $(\beta\Box_i^\bullet)$ can be applied in any derivation tree branch. Between two rules $(\alpha\Box_i^\bullet)$, $(\beta\Box_i^\bullet)$ applications at most k logical rules can be applied. Therefore, any branch in a derivation tree do not exceed height $m \cdot k$.

Derivation tree, constructed according to sequent calculus $\alpha\beta - KD45_n^\bullet$, is always less (or equal) then the one, constructed according to sequent calculus $KD45_n$. So, at least we get not worse calculus. However, new calculus has one main advantage. Sequent calculus $\alpha\beta - KD45_n^\bullet$ incorporates loop-check into inference rules. It means, that there is no need to check other sequents of the current branch. This improvement makes it to run faster.

Now we propose some lemmas to show space complexity of sequent calculus $\alpha\beta - KD45_n^\bullet$.

LEMMA 5. *Backward proof search in sequent calculus $\alpha\beta - KD45_n^\bullet$ requires at most polynomial space.*

Proof. Suppose that sequent S has length l. Every sequent in a derivation tree contains only subformulas of S (including $\Box_i^\bullet\phi$ for subformula $\Box_i\phi$). Every subformula has length $\leqslant l$ and there are $< 2 \cdot l$ different subformulas of S. We can give an index for any subformula of sequent S. We need $< 2 \cdot l \cdot l = 2 \cdot l^2$ space to store table of subformulas indexes. Every sequent in the derivation tree can be defined by two $2 \cdot l$ length arrays of subformulas indexes (one array for the left side, and one for the right side of the sequent). According to Lemma 4, height of any branch $< l^2$. Therefore, we need $< 2 \cdot l^2 + 2 \cdot 2 \cdot l \cdot l^2$ space if we use deep first search algorithm and stack. So, backward proof search in sequent calculus $\alpha\beta - KD45_n^\bullet$ requires $\mathrm{O}(l^3)$ space.

LEMMA 6. *There exist sequent with length l for which backward proof search in sequent calculus $\alpha\beta - KD45_n^\bullet$ requires $\mathrm{O}(l^3)$ space.*

Proof. We define formulas $F_1, F_2, \ldots, F_k$ as follows:
$F_1 = \neg\Box_1(\psi_1 \vee \phi_1) \vee \Box_1\phi_1$,
$F_2 = \neg\Box_1(\neg(\neg\Box_1(\psi_2 \vee \phi_2) \vee \Box_1\phi_2) \vee \phi_1) \vee \Box_1\phi_1$,
$\ldots$

$F_k = \neg\Box_1(\neg(\neg\Box_1(\ldots\neg(\neg\Box_1(\psi_k \vee \phi_k) \vee \Box_1\phi_k)\ldots \vee \phi_2) \vee \Box_1\phi_2) \vee \phi_1) \vee \Box_1\phi_1$.

Formula F_{j+1} is obtained from formula F_j by replacing subformula ψ_j with a new subformula $\neg(\neg\Box_1(\psi_{j+1} \vee \phi_{j+1}) \vee \Box_1\phi_{j+1})$.

During derivation tree construction for the sequent $\rightarrow F_k$, we can applay the following rules in the defined order (in buttom-up direction):

$(\vee R), (\neg R), (\alpha\Box_i^\bullet)$,
$(\vee L), (\neg L), (\vee R), (\neg R), (\Box^\bullet or), (\alpha\Box_i^\bullet)$,
$(\vee L), (\neg L), (\vee R), (\neg R), (\vee L), (\neg L), (\vee R), (\neg R), (\Box^\bullet or), (\alpha\Box_i^\bullet)$,
...
$(\vee L), (\neg L), (\vee R), (\neg R), \ldots, (\vee L), (\neg L), (\vee R), (\neg R), (\Box^\bullet or), (\alpha\Box_i^\bullet)$.

So, in this branch there is $3 + (1\cdot 4 + 2) + (2\cdot 4 + 2) + \ldots + ((k-1)\cdot 4 + 2) = 3 + 4\cdot\frac{k\cdot(k-1)}{2} + 2\cdot(k-1) = 2\cdot k^2 + 1$ rules applied.

Sequent $\rightarrow F_k$ has length $l = 8\cdot k$ and derivation tree height is at least $2\cdot k^2 + 1$. In other words, sequent $\rightarrow F_k$ derivation tree height is at least $2\cdot(\frac{l}{8})^2 + 1$, and maximum height is $O(l^2)$. According to Lemma 5, we use $O(l)$ space to store one sequent in the stack, and, therefore, we use $O(l^3)$ space for sequent $\rightarrow F_k$ derivation.

4. Conclusion

In this paper, we introduce sequent calculus $\alpha\beta - KD45_n^\bullet$ for multi-modal logic $KD45_n$, which uses efficient loop-check. Instead of using sequents with histories [2, 3], we use marked modal operator $\Box_i^\bullet$. For any sequent new sequent calculus constructs equal or smaller derivation tree. However, the main advantage of sequent calculus $\alpha\beta - KD45_n^\bullet$ is very restricted loop-check. During derivation tree construction we do not need to check other sequents of the current branch. This makes decision algorithm to run extremely faster.

Besides, we show the maximum height of the derivation tree and demonstrate that sequent calculus $\alpha\beta - KD45_n^\bullet$ requires polynomial ($O(l^3)$) space.

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REZIUMĖ

A. Birštunas. Efektyvus ciklų radimas multimodalinei $KD45_n$ logikai

Darbe pateiktas sekvencinis skaičiavimas multimodaliniai logikai $KD45_n$, kuris naudoja efektyvų ciklų radimo mechanizmą. Taip pat yra parodytas sekvencinio skaičiavimo $\alpha\beta - KD45_n^\bullet$ maksimalus išvedimo medžio aukštis ir sudėtingumas atminties atžvilgiu.