Liet. matem. rink, **47**, *spec. nr.*, 2007, 517–521

Multiconditional probabilities

Remigijus Petras GYLYS (MII)
e-mail: gyliene@ktl.mii.lt

Abstract. We introduce a generalization of (Kolmogorovian) conditional probabilities.

Keywords: multiconditional event, expected values, multiconditional probability.

1. Introduction

In this note we introduce multiconditional events and describe their expected values (which we call multiconditional probabilities) generalizing some results of H.T. Nguyen, I.R. Goodman and E.A. Walker [6] and partially of U. Höhle and S. Weber [1–4,7,8].

2. Multiconditional events

Let B be a Boolean algebra, and let a and f_0 be elements of B. In [5] (further developed in op.cit.), the "conditional event" "a given f_0", written $(a \parallel f_0)$, is defined as the order interval

$$(a \parallel f_0) = [a \wedge f_0, a \vee \neg f_0],$$

i.e., the set of all elements of B between $a \wedge f_0$ and $a \vee \neg f_0$ (which is the same thing as the pair $\langle a \wedge f_0, a \vee \neg f_0 \rangle$. In this note we propose the notion of "multiconditional events" in a Boolean algebra.

Let B be a Boolean algebra and let $\langle f_0, \dots, f_{n-2} \rangle$ (with $n \geqslant 2$) be a event of elements of B such that

$$f_0 \leqslant \dots \leqslant f_{n-2} \text{ and } a \wedge f_0 = \dots = a \wedge f_{n-2}.$$

Then we define the *multiconditional event* $(a \parallel f_0, \dots, f_{n-2})$ of "a given conditions $f_0, \dots, f_{n-2}$" as the following isotonic chain:

$$(a \parallel f_0, \dots, f_{n-2}) = \langle a \wedge f_0, a \vee \neg f_{n-2}, \dots, a \vee \neg f_0 \rangle.$$

We denote by $B^{\mathbf{n}}$ the set of all isotonic chains $f\colon \mathbf{n} \to B$, where $\mathbf{n}$ denotes the sequence of integers: $\mathbf{n} = \{0, 1, \dots, n-1\}$. We provide this set with the point-wise partial ordering: $f \leqslant g \Leftrightarrow f_0 \leqslant g_0, \dots, f_{n-1} \leqslant g_{n-1}$. Obviously these chains form a bounded lattice, and the lattice-theoretic operations and universal bounds are given by:

$$\langle f_0, \dots, f_{n-1} \rangle \wedge \langle g_0, \dots, g_{n-1} \rangle = \langle f_0 \wedge g_0, \dots, f_{n-1} \wedge g_{n-1} \rangle,$$

$$\langle f_0, \dots, f_{n-1}\rangle \vee \langle g_0, \dots, g_{n-1}\rangle = \langle f_0 \vee g_0, \dots, f_{n-1} \vee g_{n-1}\rangle,$$
$$\langle \bot, \dots, \bot\rangle = \bot \text{ and } \langle \top, \dots, \top\rangle = \top,$$

where $\bot$ and $\top$ (in brackets) denote the least element and the largest element in B, respectively. If we identify B with the sublattice $B_c^{\mathbf{n}}$ of $B^{\mathbf{n}}$ of constant sequences

$$B_c^{\mathbf{n}} = \{\langle a, \dots, a\rangle \mid a \in B\}$$

then B becomes a sublattice of $B^{\mathbf{n}}$.

We denote by $\tilde{B}^{\mathbf{n}}$ the set of all multiconditional events. The following equalities are worth pointing out

$$(a \wedge f_0 \parallel f_0, \dots, f_{n-2}) = \ldots = (a \wedge f_{n-2} \parallel f_0, \dots, f_{n-2}) = (a \parallel f_0, \dots, f_{n-2}).$$

It can be easy checked that chains of $B^{\mathbf{n}}$ are in an one-to-one correspondence to multiconditional events in $\tilde{B}^{\mathbf{n}}$ via

$$\langle g_0, g_1, \dots, g_{n-1}\rangle = (g_0 \parallel g_0 \vee \neg g_{n-1}, \dots, g_0 \vee \neg g_1).$$

Further, it is easy to see that the set $\tilde{B}^{\mathbf{n}}$ of multiconditional events extends the set B in the following sense:

$$\begin{aligned}
&(a \parallel \top, \dots, \top) = \langle a, \dots, a\rangle,\\
&(a \parallel f_0, \dots, f_{n-2}) = (b \parallel g_0, \dots, g_{n-2})\\
&\quad \Leftrightarrow a \wedge f_0 = b \wedge g_0,\ f_0 = g_0, \dots, f_{n-2} = g_{n-2}.
\end{aligned}$$

The following natural partial ordering can be defined on $\tilde{B}^{\mathbf{n}}$:

$$\begin{aligned}
&(a \parallel f_0, \dots, f_{n-2}) \leqslant (b \parallel g_0, \dots, g_{n-2})\\
&\quad \Leftrightarrow a \wedge f_0 \leqslant b \wedge g_0, a \vee \neg f_0 \leqslant b \vee \neg g_0, \dots, a \vee \neg f_{n-2} \leqslant b \vee \neg g_{n-2},
\end{aligned}$$

which extends the usual entailment relation in B. The following monotonicity properties hold:

$$\begin{aligned}
&a \leqslant b \Rightarrow (a \parallel f_0, \dots, f_{n-2}) \leqslant (b \parallel f_0, \dots, f_{n-2}),\\
&f_0 \leqslant g_0, \dots, f_{n-2} \leqslant g_{n-2}, a \wedge g_0 \leqslant a \wedge f_0\\
&\quad \Rightarrow (a \parallel g_0, \dots, g_{n-2}) \leqslant (a \parallel f_0, \dots, f_{n-2}).
\end{aligned}$$

Furthermore, $\tilde{B}^{\mathbf{n}}$ is a bounded lattice. The lattice operations and universal bounds are given by

$$\begin{aligned}
&(a \parallel f_0, \dots, f_{n-2}) \wedge (b \parallel g_0, \dots, g_{n-2})\\
&\quad = \big(a \wedge f_0 \wedge b \wedge g_0 \parallel (a \wedge f_0 \wedge b \wedge g_0) \vee (\neg a \wedge f_0) \vee (\neg b \wedge g_0),\\
&\qquad \dots, (a \wedge f_0 \wedge b \wedge g_0) \vee (\neg a \wedge f_{n-2}) \vee (\neg b \wedge g_{n-2})\big),\\
&(a \parallel f_0, \dots, f_{n-2}) \vee (b \parallel g_0, \dots, g_{n-2})
\end{aligned}$$

$$= \big((a \wedge f_0) \vee (b \wedge g_0) \parallel (a \wedge f_0) \vee (b \wedge g_0) \vee (\neg a \wedge f_0 \wedge \neg b \wedge g_0),$$
$$\dots, (a \wedge f_0) \vee (b \wedge g_0) \vee (\neg a \wedge f_{n-2} \wedge \neg b \wedge g_{n-2})\big),$$
$$\tilde{\bot} = (\bot \parallel \top, \dots, \top) \text{ and } \tilde{\top} = (\top \parallel \top, \dots, \top).$$

If we identify B with the sublattice $\tilde{B}_c^{\mathbf{n}}$ of $\tilde{B}^{\mathbf{n}}$ of multiconditional events

$$\tilde{B}_c^{\mathbf{n}} = \{(a \parallel \top \dots \top) \mid a \in B\}$$

then B becomes a sublattice of $\tilde{B}^{\mathbf{n}}$.

3. Multiconditional probabilities

Let P be a probability measure on a Boolean algebra B. If $\langle g_0, \dots, g_{n-1}\rangle \subseteq B$ is an isotonic chain, then from isotonicity of P it follows that the image $P\langle g_0, \dots, g_{n-1}\rangle$ of $\langle g_0, \dots, g_{n-1}\rangle$ under P

$$P\langle g_0, \dots, g_{n-1}\rangle = \langle P(g_0), \dots, P(g_{n-1})\rangle$$

is an isotonic chain in the real unit interval $[0, 1]$ (with the 'natural' lattice structure given by max and min). For $(a \parallel f_0, \dots, f_{n-2}) \in \tilde{B}^{\mathbf{n}}$, we have that

$$\begin{aligned} P(a \parallel f_0, \dots, f_{n-2}) &= \langle P(a \wedge f_0), P(a \vee \neg f_{n-2}), \dots, P(a \vee \neg f_0)\rangle \\ &= \big\langle P(a \wedge f_0), P(a \wedge f_0) + 1 - P(f_{n-2}), \dots, P(a \wedge f_0) + 1 - P(f_0)\big\rangle. \quad (1) \end{aligned}$$

Consider an "expected value" function E on $[0, 1]$, a n-ary function $E\colon [0, 1]^{\mathbf{n}} \to [0, 1]$ satisfying the following axioms: for $r \in [0, 1]$, $\langle r_0, \dots, r_{n-1}\rangle, \langle q_0, \dots, q_{n-1}\rangle \in [0, 1]^{\mathbf{n}}$ (with $r_0 \leqslant \dots \leqslant r_{n-1}$ and $q_0 \leqslant \dots \leqslant q_{n-1}$),

(i) $E\langle r, \dots, r\rangle = r$ (idempotency),
(ii) $r_0 \leqslant q_0, \dots, r_{n-1} \leqslant q_{n-1} \Rightarrow E\langle r_0, \dots, r_{n-1}\rangle \leqslant E\langle q_0, \dots, q_{n-1}\rangle$ (isotonicity).

Now we are going to "extend" the probability measure P to the lattice $\tilde{B}^{\mathbf{n}}$ of multiconditional events in the following way:

$$\begin{aligned} (a \parallel f_0, \dots, f_{n-2}) &\mapsto E\big(P(a \parallel f_0, \dots, f_{n-2})\big) \\ &= E\big\langle P(a \wedge f_0), P(a \wedge f_0) + 1 - P(f_{n-2}), \dots, P(a \wedge f_0) + 1 - P(f_0)\big\rangle. \end{aligned}$$

We denote the values of this extension of P as $P_E(a \mid f_0, \dots, f_{n-2})$ and call it *multiconditional probability* (of "a given conditions $f_0, \dots, f_{n-2}$"). Obviously this quantity satisfies the following conditions:

(i) $P_E(\bot \mid \top, \dots, \top) = 0$ and $P_E(\top \mid \top, \dots, \top) = 1$,
(ii) $(a \parallel f_0, \dots, f_{n-2}) \leqslant (b \parallel g_0, \dots, g_{n-2})$
$\Rightarrow P_E(a \mid f_0, \dots, f_{n-2}) \leqslant P_E(b \mid g_0, \dots, g_{n-2})$.

To motivate the choice of the name "multiconditional probability", consider an expected value function defined by

$$E_{2,k}\langle r_0, r_1\rangle = \begin{cases} \frac{r_0}{r_0+1-r_1} & \text{if } \langle r_0, r_1\rangle \neq \langle 0, 1\rangle, \\ k & \text{if } \langle r_0, r_1\rangle = \langle 0, 1\rangle, \end{cases}$$

where k is an arbitrary number in $[0,1]$. From (1) (with $n=2$) it follows that

$$P_{E_{2,k}}(a \mid f_0) = \begin{cases} \frac{P(a\wedge f_0)}{P(a\wedge f_0)+1-(P(a\wedge f_0)+1-P(f_0))} & \text{if } P(f_0) \neq 0, \\ k & \text{if } P(f_0) = 0 \end{cases}$$

$$= \begin{cases} \frac{P(a\wedge f_0)}{P(f_0)} & \text{if } P(f_0) \neq 0, \\ k & \text{if } P(f_0) = 0, \end{cases}$$

which (in the case when $k=1$) is the usual definition of (Kolmogorovian) conditional probability.

For $\langle r_0, r_1, r_2\rangle \in [0,1]^{\mathbf{3}}$ (with $r_0 \leqslant r_1 \leqslant r_2$) and $k \in [0,1]$, consider

$$E_{3,k}\langle r_0, r_1, r_2\rangle = \begin{cases} \frac{r_0}{r_0+1-\frac{r_1}{r_1+1-r_2}} & \text{if } \langle r_0, r_2\rangle \neq \langle 0,1\rangle, \\ k & \text{if } \langle r_0, r_2\rangle = \langle 0,1\rangle, \end{cases}$$

which defines an expected value function from $[0,1]^{\mathbf{3}}$ to $[0,1]$. From (1) (with $n=3$) it follows that

$$P_{E_{3,k}}(a \mid f_0, f_1) = \begin{cases} 1 - \frac{P(f_0)-P(a\wedge f_0)}{P(f_0)-P(a\wedge f_0)(P(f_1)-P(f_0))} & \text{if } P(f_0) \neq 0, \\ k & \text{if } P(f_0) = 0 \end{cases}$$

(not forgetting the conditions: $f_0 \leqslant f_1$ and $a \wedge f_0 = a \wedge f_1$). This formula can be considered as a generalization of the usual conditional probability.

Next, for $\langle r_0, r_1, r_2, r_3\rangle \in [0,1]^{\mathbf{4}}$ (with $r_0 \leqslant \ldots \leqslant r_3$) and $k \in [0,1]$, consider

$$E_{4,k}\langle r_0, r_1, r_2, r_3\rangle = \begin{cases} \frac{r_0}{r_0+1-\frac{r_1}{r_1+1-\frac{r_2}{r_2+1-r_3}}} & \text{if} langle\ r_0, r_3\rangle \neq \langle 0,1\rangle, \\ k & \text{if } \langle r_0, r_3\rangle = \langle 0,1\rangle. \end{cases}$$

It is evident that this quantity defines an expected value function from $[0,1]^{\mathbf{4}}$ to $[0,1]$. From this we obtain that

$$P_{E_{4,k}}(a \mid f_0, f_1, f_2)$$
$$= \begin{cases} 1 - \frac{P(f_0)-P(a\wedge f_0)}{P(f_0)-P(a\wedge f_0)((P(f_1)-P(f_0))(1-P(f_2)+P(a\wedge f_0))+P(f_2)-P(f_0))} & \text{if } P(f_0)\neq 0, \\ k & \text{if } P(f_0)=0 \end{cases}$$

(with the conditions that $f_0 \leqslant f_1 \leqslant f_2$ and $a \wedge f_0 = a \wedge f_1 = a \wedge f_2$), which can be considered as an another (more high level) generalization of the traditional conditional probability.

Similarly, one can consider the case $n=5$ etc.

References

1. U. Höhle, Commutative, residuated l-monoids, in: U. Höhle and E.P. Klement (Eds.), *Nonclassical Logics and Their Applications to Fuzzy Subsets*, Kluwer Academic Publishers, Boston, Dordrecht (1995), pp. 53–106.

2. U. Höhle, MV-algebra valued filter theory, *Questiones Math.*, **19**, 23–46 (1996).
3. U. Höhle, S. Weber, Uncertainty measures, realizations and entropies, in: J. Gontsias, R.P.S. Mahler, H.T. Nguyen (Eds.), *Random Sets*: *Theory and Applications*, Springer-Verlag, Heidelberg/Berlin/New York (1997), pp. 259–295.
4. U. Höhle, S. Weber, On conditioning operators, in: U. Höhle, S.E. Rodabaugh (Eds.), *Mathematics of Fuzzy Sets*, Kluwer Academic Publishers, Boston/Dordrecht/London (1999), pp. 653–673.
5. B.O. Koopman, The axioms and algebra of intuitive probability, *Ann. Math.*, **41**, 269–292 (1940).
6. H.T. Nguyen, I.R. Goodman and E.A. Walker, *Conditional Inference and Logic for Intelligent Systems*: *A Theory of Measure-Free Conditioning*, North-Holland, Amsterdam (1991).
7. S. Weber, Conditioning on MV-algebras and additive measures, Part I, *Fuzzy Sets and Systems*, **92**, 241–250 (1997).
8. S. Weber, Conditioning on MV-algebras and additive measures – further results, in: D. Dubois, H. Prade, E.P. Klement (Eds.), *Fuzzy Sets, Logics and Reasoning about Knowledges*, Kluwer Academic Publishers, Boston/Dordrecht (1999), pp. 175–199.

REZIUMĖ

R.P. Gylys. Daugiasąlyginės tikimybės

Pristatoma ir pailiustruojama pavyzdžiais nauja daugiasąlyginių tikimybių sąvoka.