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On the argument of zeta-functions of certain cusp forms

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Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, integers, real and complex numbers, respectively. The set

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \colon a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is called the full modular group. Suppose that the function $F(z)$ is holomorphic in the upper half – plane $\operatorname{Im} z > 0$, with some κ, for all elements $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ of $SL(2,\mathbb{Z})$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z)$$

and $\lim_{\operatorname{Im} z \to \infty} F(z) = 0$. Then $F(z)$ is called a cusp form of weight κ for the full modular group. We also suppose that $F(z)$ is a normalized eigenform for all Hecke operators. In this case, $F(z)$ has the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi i m z}, \quad c(1) = 1.$$

Let $s = \sigma + it$ be a complex variable. Then the function

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

is called the zeta-function attached to the cusp form $F(z)$. The arithmetical function $c(m)$ is multiplicative. Therefore, $\varphi(s, F)$ has the Euler product expansion over primes

$$\varphi(s, F) = \prod_{p} \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $c(p) = \alpha(p) + \beta(p)$. In view of Deligne's estimates [1]

$$|\alpha(p)| \leqslant p^{\frac{\kappa-1}{2}}, \quad |\beta(p)| \leqslant p^{\frac{\kappa-1}{2}}$$

it is easily seen that both the Euler product and the Dirichlet series for $\varphi(s, F)$ converge absolutely for $\sigma > (\kappa + 1)/2$, and in this region $\varphi(s, F)$ is a non-vanishing holomorphic function. Moreover, $\varphi(s, F)$ is analytically continuable to an entire function and satisfies the following functional equation

$$(2\pi)^{-s}\Gamma(s)\varphi(s, F) = (-1)^{\kappa/2}(2\pi)^{s-\kappa}\Gamma(\kappa - s)\varphi(\kappa - s, F),$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function. The nontrivial zeros of $\varphi(s, F)$ lie in the critical strip $\{s \in \mathbb{C}\colon\ (\kappa - 1)/2 < \sigma < (\kappa + 1)/2\}$. The analogue of the Riemann hypothesis says that all nonntrivial zeros of $\varphi(s, F)$ lie on the critical line $\sigma = \kappa/2$.

The aim of this note is to obtain a formula for $\arg(\varphi(s, F))$ near the critical line. We will use the notation

$$\sigma_T = \frac{\kappa}{2} + \frac{1}{l_T},$$

where $l_T \to \infty$ as $T \to \infty$.

Taking logarithms and differentiating both sides of the Euler product with respect to s, we find that, for $\sigma > (\kappa + 1)/2$,

$$-\frac{\varphi'}{\varphi}(s, F) = \sum_p \sum_{m=1}^{\infty} \big(\alpha^m(p) + \beta^m(p)\big) p^{-ms} \log p. \tag{1}$$

Now we define

$$\Lambda_F(n) = \begin{cases} \big(\alpha^m(p) + \beta^m(p)\big) \log p, & \text{if } n = p^m, \\ 0, & \text{otherwise.} \end{cases}$$

Then in view of (1)

$$-\frac{\varphi'}{\varphi}(s, F) = \sum_{n=2}^{\infty} \Lambda_F(n) n^{-s}, \quad \sigma > \frac{\kappa + 1}{2}.$$

Let for $x > 1$,

$$\Lambda_{x,F}(n) = \begin{cases} \Lambda_F(n), & \text{if } 1 \leqslant n \leqslant x, \\ \Lambda_F(n) \frac{(\log(x^3/n))^2 - 2(\log(x^2/n))^2}{2(\log x)^2}, & \text{if } x \leqslant n \leqslant x^2, \\ \Lambda_F(n) \frac{(\log(x^3/n))^2}{2(\log x)^2}, & \text{if } x^2 \leqslant n \leqslant x^3. \end{cases} \tag{2}$$

THEOREM. *For $t \geqslant 2$, $2 \leqslant x \leqslant t^2$, we have*

$$\arg \varphi(\sigma_T + it, F) = -\sum_{n < x^3} \frac{\Lambda_{x,F}(n) \sin(t \log n)}{n^{\sigma_{T,x,t}} \log n}$$
$$+ \mathrm{O}\bigg((\sigma_{T,x,t} - \sigma_T) \bigg| \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}}} \bigg| \bigg) + \mathrm{O}\big((\sigma_{T,x,t} - \sigma_T) \log t \big),$$

where

$$\sigma_{T,x,t} = \sigma_T + 2\max(\beta - \kappa/2,\ \ 2/\log x) \tag{3}$$

with $\rho = \beta + i\gamma$ *running over those zeros* $\varphi(s, F)$ *for which*

$$|t - \gamma| \leqslant x^{3|\beta-\kappa/2|}(\log x)^{-1}, \tag{4}$$

and $\Lambda_{x,F}(n)$ *is as in* (2).

For the proof of the theorem we will apply two following lemmas.

LEMMA 1. *If* $s \neq \rho, t \geqslant 2$, *then*

$$\frac{\varphi'}{\varphi}(s, F) = \sum_{\rho}\big((s-\rho)^{-1} + \rho^{-1}\big) + \mathrm{O}(\log t)$$

uniformly for $\kappa/2 \leqslant \sigma \leqslant \kappa/2 + 10$.

Proof of this lemma can be found in [3].

LEMMA 2. *Let* $t \geqslant 2$ *and* $2 \leqslant x \leqslant t^2$. *Then we have that*

$$\sum_{\rho} \frac{\sigma_{T,x,t} - \frac{\kappa}{2}}{(\sigma_{T,x,t} - \beta)^2 + (t-\beta)^2} = \mathrm{O}\bigg|\sum_{m<x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}+it}}\bigg| + \mathrm{O}(\log t)$$

and

$$\begin{aligned}\frac{\varphi'}{\varphi}(s, F) = &-\sum_{n<x^3} \frac{\Lambda_{x,F}(n)}{n^s}\\ &+ \mathrm{O}\bigg((x^{\sigma_T/2-\sigma/2})\bigg|\sum_{n<x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}+it}}\bigg|\bigg) + \mathrm{O}(x^{\sigma_T/2-\sigma/2}\log t)\end{aligned} \tag{5}$$

for $\sigma > \sigma_{T,x,t}$.

Proof of this lemma is similar to the proof of Theorem 3.5.2 from [2].

Proof of the Theorem. Clearly,

$$\begin{aligned}\arg\varphi(\sigma_T + it) &= -\int_{\sigma_T}^{\infty} \mathrm{Im}\,\frac{\varphi'}{\varphi}(\sigma + it)\,\mathrm{d}\sigma\\ &= \int_{\sigma_{T,x,t}}^{\infty} \mathrm{Im}\,\frac{\varphi'}{\varphi}(\sigma + it)\,\mathrm{d}\sigma + (\sigma_{T,x,t} - \sigma_T)\mathrm{Im}\,\frac{\varphi'}{\varphi}(\sigma_{T,x,t} + it)\\ &\quad + \int_{\sigma_T}^{\sigma_{T,x,t}} \mathrm{Im}\bigg\{\frac{\varphi'}{\varphi}(\sigma_{T,x,t} + it) - \frac{\varphi'}{\varphi}(\sigma + it)\bigg\}\,\mathrm{d}\sigma\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{6}$$

In view of estimate (5)

$$I_1 = \operatorname{Im} \sum_{n<x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}+it}\log n} + \mathrm{O}\Big(\frac{1}{\log x}\Big|\sum_{n<x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}+it}}\Big|\Big) + \mathrm{O}\Big(\frac{\log t}{\log x}\Big). \tag{7}$$

Similarly as in the proof of Lemma 3.6.3 from [2], we find that

$$I_2 = \mathrm{O}\bigg((\sigma_{T,x,t}-\sigma_T)\Big|\sum_{m<x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}+it}}\Big|\bigg) + \mathrm{O}\big((\sigma_{T,x,t}-\sigma_T)\ln t\big). \tag{8}$$

From Lemma 1, taking the imaginary part of both sides, it follows that (we suppose for simplicity that $t \neq \gamma$)

$$\begin{aligned}
&\operatorname{Im}\Big(\frac{\varphi'}{\varphi}(\sigma_{T,x,t}+it) - \frac{\varphi'}{\varphi}(\sigma+it)\Big) \\
&\quad = \sum_{\rho} \operatorname{Im}\Big(\frac{1}{\sigma_{T,x,t}-\beta+i(t-\gamma)} - \frac{1}{\sigma-\beta+i(t-\gamma)}\Big) + \mathrm{O}(\log t) \\
&\quad = \sum_{\rho} \frac{(\sigma_{T,x,t}-\sigma)(\sigma+\sigma_{T,x,t}-2\beta)(t-\gamma)}{((\sigma_{T,x,t}-\beta)^2+(t-\gamma)^2)((\sigma-\beta)^2+(t-\gamma)^2)} + \mathrm{O}(\log t)
\end{aligned}$$

for $\kappa/2 \leqslant \sigma \leqslant \sigma_{T,x,t}$. Therefore,

$$\begin{aligned}
|I_3| &\leqslant \sum_{\rho} \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{(\sigma_{T,x,t}-\sigma)|\sigma+\sigma_{T,x,t}-2\beta||t-\gamma|}{((\sigma_{T,x,t}-\beta)^2)+(t-\gamma)^2((\sigma-\beta)^2)+(t-\gamma)^2}\,\mathrm{d}\sigma \\
&\qquad + \mathrm{O}\big((\sigma_{T,x,t}-\sigma_T)\log t\big) \\
&\leqslant \sum_{\rho} \frac{\sigma_{T,x,t}-\sigma_T}{(\sigma_{T,x,t}-\beta)^2+(t-\gamma)^2} \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma+\sigma_{T,x,t}-2\beta||t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma \\
&\qquad + \mathrm{O}\big((\sigma_{T,x,t}-\sigma_T)\log t\big).
\end{aligned} \tag{9}$$

First, let

$$|\beta-\kappa/2| \leqslant \frac{1}{2}(\sigma_{T,x,t}-\kappa/2).$$

Then

$$|\sigma+\sigma_{T,x,t}-2\beta| \leqslant (\sigma-\kappa/2)+(\sigma_{T,x,t}-\kappa/2)+2|\beta-\kappa/2| \leqslant 3(\sigma_{T,x,t}-\kappa/2)$$

for $\sigma_T \leqslant \sigma \leqslant \sigma_{T,x,t}$. Thus

$$\begin{aligned}
&\int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma+\sigma_{T,x,t}-2\beta||t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma \leqslant 3(\sigma_{T,x,t}-\kappa/2)\int_{-\infty}^{\infty} \frac{|t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma \\
&\quad = 3(\sigma_{T,x,t}-\kappa/2)\int_{-\infty}^{\infty} \frac{|t-\gamma|}{y^2+(t-\gamma)^2}\,\mathrm{d}y = 3\pi(\sigma_{T,x,t}-\kappa/2).
\end{aligned} \tag{10}$$

If
$$|\beta-\kappa/2| > \frac{\sigma_{T,x,t}-\kappa/2}{2} \geqslant \frac{\sigma_{T,x,t}-\sigma_T}{2},$$
then if follows from (3) and (4) that
$$|t-\gamma| > \frac{x^{3|\beta-\kappa/2|}}{\log x} > 3|\beta-\kappa/2|,$$
and
$$|\sigma+\sigma_{T,x,t}-2\beta| \leqslant (\sigma-\kappa/2)+(\sigma_{T,x,t}-\kappa/2)+2|\beta-\kappa| < 6|\beta-\kappa/2|$$
for $\sigma_T \leqslant \sigma \leqslant \sigma_{T,x,t}$. Therefore, in this case, too
$$\int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma+\sigma_{T,x,t}-2\beta||t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma < \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma+\sigma_{T,x,t}-2\beta|\,\mathrm{d}\sigma}{|t-\gamma|}$$
$$< 2\frac{|\beta-\kappa/2|}{|\beta-\kappa/2|}\int_{\sigma_T}^{\sigma_{T,x,t}} \mathrm{d}\sigma < 10(\sigma_{T,x,t}-\sigma_T) < 10(\sigma_{T,x,t}-\kappa/2).$$
This, (7), (8) and Lemma 2 yield
$$|I_3| < 10(\sigma_{T,x,t}-\sigma)\sum_{\rho}\frac{\sigma_{T,x,t}-\kappa/2}{(\sigma_{T,x,t}-\beta)^2+(t-\gamma)^2} + \mathrm{O}\big((\sigma_{T,x,t}-\sigma_T)\log t\big)$$
$$= \mathrm{O}\Bigg((\sigma_{T,x,t}-\sigma_T)\Bigg|\sum_{n<x^3}\frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t}+it}}\Bigg|\Bigg) + \mathrm{O}\big((\sigma_{T,x,t}-\sigma_T)\log t\big). \tag{11}$$
Now, substituting the estimates (7), (8) and (11), we obtain the assertion of the theorem.

References

1. P. Deligne, La conjecture de Weil I, *Pub. I. H. E. S.*, **43**, 273–307 (1974).
2. A. Laurinčikas, *Limit Theorems for the Riemann Zeta-function*, Kluwer, Dordrecht, Boston, London (1996).
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REZIUMĖ

R. Ivanauskaitė. Parabolinių formų dzeta funkcijos argumentas

Darbe įrodyta parabolinių formų dzeta funkcijos teorema argumentui.