Lietuvos matematikos rinkinys
Proc. of the Lithuanian Mathematical Society, Ser. A
Vol. 62, 2021, pages 1–8
https://doi.org/10.15388/LMR.2021.25128

Relations between spectrum curves of discrete Sturm–Liouville problem with nonlocal boundary conditions and graph theory. II

Jonas Vitkauskas, Artūras Štikonas

Institute of Applied Mathematics, Vilnius University
Naugarduko 24, LT-03225 Vilnius
E-mail: jonas.vitkauskas@mif.stud.vu.lt; arturas.stikonas@mif.vu.lt

Received July 23, 2021; published online December 1, 2021

Abstract. In this paper, relations between discrete Sturm–Liouville problem with nonlocal integral boundary condition characteristics (poles, critical points, spectrum curves) and graphs characteristics (vertices, edges and faces) were found. The previous article was devoted to the Sturm–Liouville problem in the case two-points nonlocal boundary conditions.

Keywords: Sturm–Liouville problem; spectrum curves; integral boundary condition; graphs

AMS Subject Classification: 34B24; 05C10

1 A discrete Sturm–Liouville Problem

In this paper, particular properties of the spectrum of *a discrete Sturm–Liouville Problem* (dSLP) [3, 4, 5] with *Integral Boundary Condition* were found using Euler's charakteristic formula [2]. In previous article [5] we have found relations between spectrum curve properties and graphs theory in the case of two-points nonlocal boundary conditions.

We introduce a uniform grid and we use notation $\bar{\omega}^h = \{t_j = jh, j = 0, \ldots, n; nh = 1\}$ for $2 < n \in \mathbb{N}$, and $\mathbb{N}^h := (0, n) \cap \mathbb{N}$, $\overline{\mathbb{N}}^h = \mathbb{N}^h \cup \{0, n\}$. Also, we make an assumption that ξ_1 and ξ_2 are located on the grid, i.e., $\xi_1 = m_1 h = m_1/n$, $\xi_2 = m_2 h = m_2/n$, $\boldsymbol{m} \in S^h_{\boldsymbol{\xi}} := \{(m_1, m_2) : 0 \le m_1 < m_2 \le n, m_1, m_2 \in \overline{\mathbb{N}}^h\}$. Let $\boldsymbol{\xi} = \boldsymbol{m}/n = (m_1/n, m_2/n)$, $\xi = \xi_1/\xi_2 = m_1/m_2$, $\xi_+ = \xi_1 + \xi_2 = m_+/n$,

© 2021 Authors. Published by Vilnius University Press
This is an Open Access article distributed under the terms of the Creative Commons Attribution Licence, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

$\xi_- = \xi_2 - \xi_1 = m_-/n$, here $m_+ := m_1 + m_2$, $m_- = m_2 - m_1$. We denote $\mathbb{N}_{odd} = \{k \in \mathbb{N}\colon k - \text{odd}\}$, $\mathbb{N}_{even} = \{k \in \mathbb{N}\colon k - \text{even}\}$.

Let us consider a dSLP (an approximation by Finite-Difference Scheme) [3, 5]

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = \lambda U_i, \qquad i = 1, \dots, n-1, \tag{1}$$

$\lambda \in \mathbb{C}$ with a classical discrete Dirichlet Boundary Condition (BC)

$$U_0 = 0 \tag{2}$$

and Integral Boundary Condition (approximated by trapezoidal formula):

$$U_n = \gamma h\left(\frac{U_{m_1} + U_{m_2}}{2} + \sum_{k=m_1+1}^{m_2-1} U_k\right). \tag{3}$$

Let us consider a bijection (see [1])

$$\lambda = \lambda^h(q) := \frac{4}{h^2} \sin^2 \frac{\pi q h}{2} \tag{4}$$

between $\mathbb{C}_\lambda := \mathbb{C}$ and $\mathbb{C}_q^h$, $\mathbb{C}_q^h := \mathbb{R}_q^h \cup \mathbb{C}_q^{h+} \cup \mathbb{C}_q^{h-}$, $\mathbb{R}_q^h := \mathbb{R}_y^- \cup \{0\} \cup \mathbb{R}_x^h \cup \{n\} \cup \mathbb{R}_y^{h+}$, $\mathbb{R}_y^- := \{q = \imath y : y > 0\}$, $\mathbb{R}_x^h := \{q = x : 0 < x < n\}$, $\mathbb{R}_y^{h+} := \{q = n + \imath y : y > 0\}$, $\mathbb{C}_q^{h+} := \{q = x + \imath y : 0 < x < n, y > 0\}$, $\mathbb{C}_q^{h-} := \{q = x + \imath y : 0 < x < n, y < 0\}$. The general solution U_j for a discrete equation (1) is equal to:

$$U_j = C_1 \sin(\pi q t_j)(1 - hq)^{-1}\pi^{-1}q^{-1} + C_2 \cos(\pi q t_j).$$

Then by using BCs (2) and (3) we get an equation:

$$Z^h(q) = \gamma P_{\boldsymbol{\xi}}^h(q), \quad q \in \mathbb{C}_q^h,$$

where functions $Z^h(q)$ and $P_{\boldsymbol{\xi}}^h(q)$ are as follows:

$$Z^h(q) = \frac{\sin(\pi q)}{\pi q} \cdot \frac{\sin(\frac{\pi}{2} q h)}{\pi q h \cos(\frac{\pi}{2} q h)}, \quad P_{\boldsymbol{\xi}}^h(q) = \frac{\sin\left(\frac{\pi}{2} q(\xi_2 - \xi_1)\right)}{\pi q} \cdot \frac{\sin\left(\frac{\pi}{2} q(\xi_2 + \xi_1)\right)}{\pi q}$$

Constant Eigenvalues. For any constant eigenvalue $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system [1]:

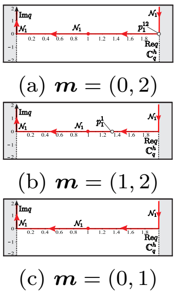
$$Z^h(q) = 0, \quad P_{\boldsymbol{\xi}}^h(q) = 0.$$

For every CEP c_j we define *nonregular Spectrum Curve* $\mathcal{N}_j = \{c_j\}$.

Nonconstant eigenvalues. Let us consider *Complex Characteristic Function*:

$$\gamma_c(q) = \gamma_c(q; \boldsymbol{\xi}) := \frac{Z^h(q)}{P_{\boldsymbol{\xi}}^h(q)}, \qquad q \in \mathbb{C}_q^h. \tag{5}$$

All *nonconstant eigenvalues* (which depend on the parameter γ) are γ-points of (Complex-Real) *Characteristic Function* (CF)[6]. CF $\gamma(q)$ is the restriction of Complex CF $\gamma_c(q)$ on a set $\mathcal{D}_{\boldsymbol{\xi}} := \{q \in \mathbb{C}_q^h\colon \operatorname{Im}\gamma_c(q) = 0\}$ (see more in [5]). We call such

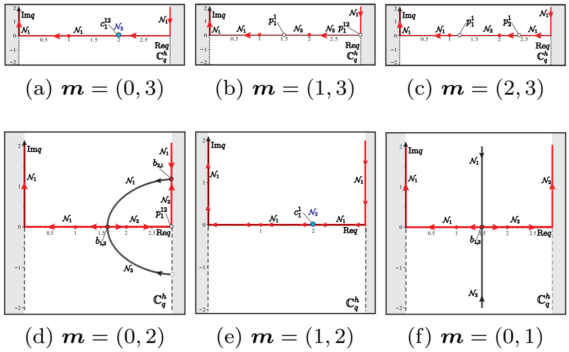


(a) $\boldsymbol{m} = (0, 2)$

(b) $\boldsymbol{m} = (1, 2)$

(c) $\boldsymbol{m} = (0, 1)$

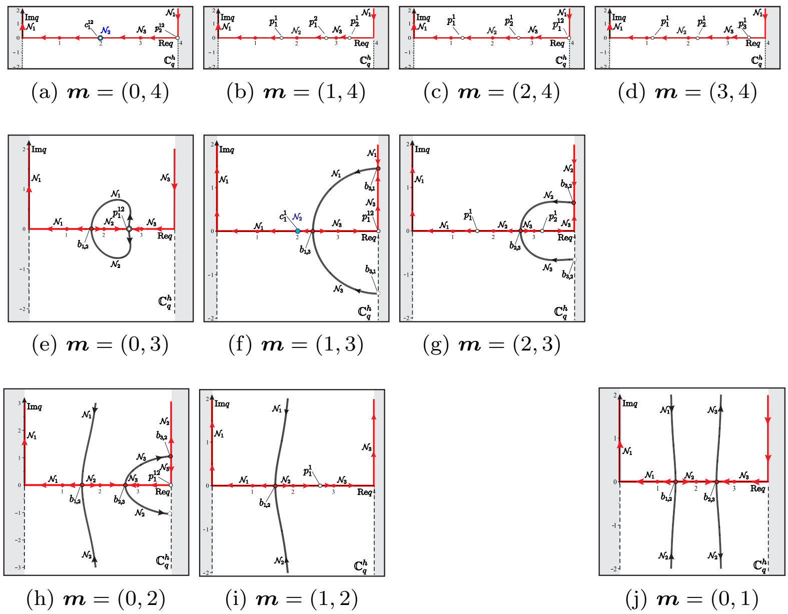
Fig. 1. Spectrum Curves for $n = 2$[4].



(a) $\boldsymbol{m} = (0, 3)$ (b) $\boldsymbol{m} = (1, 3)$ (c) $\boldsymbol{m} = (2, 3)$

(d) $\boldsymbol{m} = (0, 2)$ (e) $\boldsymbol{m} = (1, 2)$ (f) $\boldsymbol{m} = (0, 1)$

Fig. 2. Spectrum Curves for $n = 3$ [4].



(a) $\boldsymbol{m} = (0, 4)$ (b) $\boldsymbol{m} = (1, 4)$ (c) $\boldsymbol{m} = (2, 4)$ (d) $\boldsymbol{m} = (3, 4)$

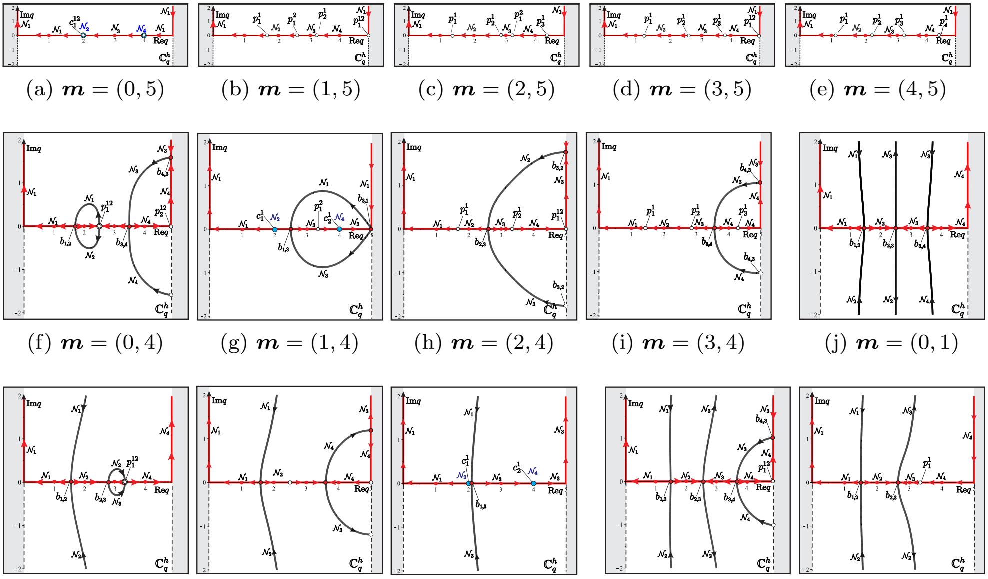
(e) $\boldsymbol{m} = (0, 3)$ (f) $\boldsymbol{m} = (1, 3)$ (g) $\boldsymbol{m} = (2, 3)$

(h) $\boldsymbol{m} = (0, 2)$ (i) $\boldsymbol{m} = (1, 2)$ (j) $\boldsymbol{m} = (0, 1)$

Fig. 3. Spectrum Curves for $n = 4$ [4].

curves *regular Spectrum Curves* [1, 4]. The regular Spectrum Curves form Spectrum Domain in $\mathbb{C}_q^h \cup \{\infty\}$ (see Figures 1–5 for the first n).

Each regular Spectrum Curve begins at the pole point ($\gamma = -\infty$) of CF and ends at the pole point ($\gamma = +\infty$) of CF. We denote a set Poles $\mathcal{P} := \{p_i, i = \overline{1, \bar{n}_p}\}$, where $\bar{n}_p$ is the number of poles at $\mathbb{C}_q^h$. For our problems $\mathcal{P} \subset \mathbb{R}_x^h \cup \{n\}$ and all such poles are of the first order or the second order. We denote n_{1p}, n_{2p} the number of the first order and the second order poles, respectively. We use notation $n_p := n_{1p} + 2n_{2p}$ for number poles with multiplicity. So, $\bar{n}_p = n_{1p} + n_{2p} = n_p - n_{2p}$.



(a) $\boldsymbol{m} = (0,5)$ (b) $\boldsymbol{m} = (1,5)$ (c) $\boldsymbol{m} = (2,5)$ (d) $\boldsymbol{m} = (3,5)$ (e) $\boldsymbol{m} = (4,5)$

(f) $\boldsymbol{m} = (0,4)$ (g) $\boldsymbol{m} = (1,4)$ (h) $\boldsymbol{m} = (2,4)$ (i) $\boldsymbol{m} = (3,4)$ (j) $\boldsymbol{m} = (0,1)$

(k) $\boldsymbol{m} = (0,3)$ (l) $\boldsymbol{m} = (1,3)$ (m) $\boldsymbol{m} = (2,3)$ (n) $\boldsymbol{m} = (0,2)$ (o) $\boldsymbol{m} = (1,2)$

Fig. 4. Spectrum Curves for $n = 5$ [4].

There exist Pole Points (PP) of the second order. They are described as follows:

$$p_k^{12} = 2nk/\gcd(m_+, m_-), \quad k = 1, \ldots, \lfloor n/p_1^{12} \rfloor .$$

All PP of the second order are in $\mathbb{R}_x^h$. Note that there could be a PP of second order in Ramification Point $q = n$, so they become simple PP of the first order.

Two or more Spectrum Curves may intersect at CP. We denote a set CP $\mathcal{B} := \{b_i, i = \overline{1, n_b}\}$, where n_b is the number of CPs at $\mathbb{C}_q^h$. The number of CPs at $\mathbb{R}_q^h$ and $\mathbb{C}_q^{h+}$ we denote as n_{cr} and n_{cr}^+, respectively. Note that the part of the spectrum domain in set $\mathbb{C}_q^{h+}$ is symmetric to the part in set $\mathbb{C}_q^{h-}$. So, $n_b = n_{cr} + 2n_{cr}^+$. If $b \in \mathcal{B}$ then $\deg^+(b)$ is one unit larger than the order of this CP.

The pole at $q = \infty$ is of

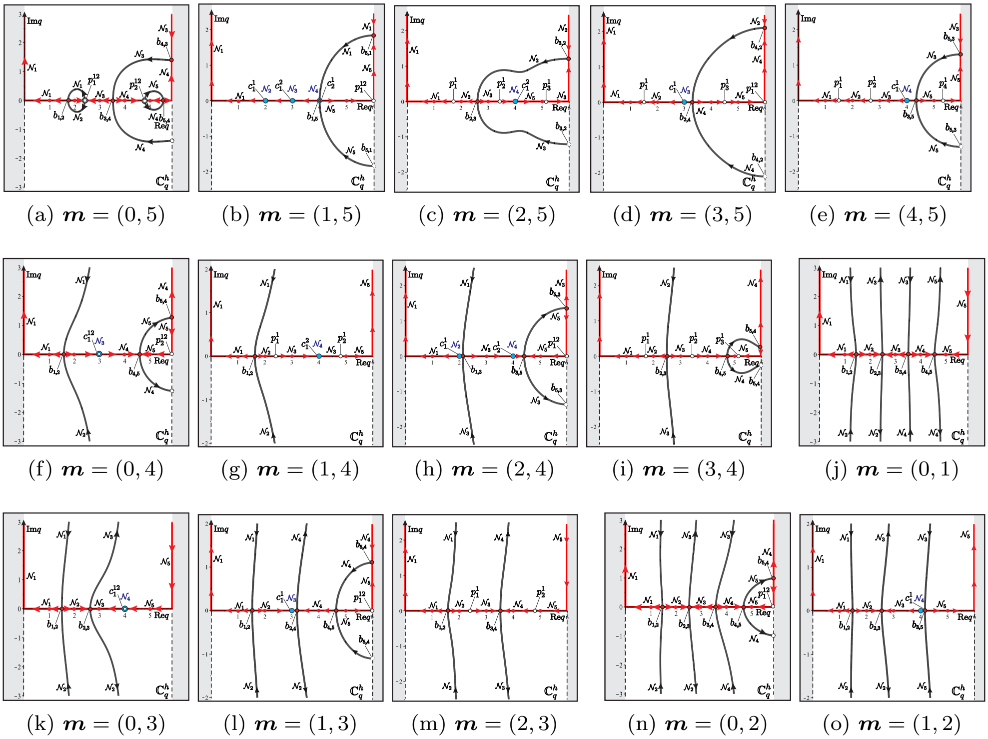
$$n_\infty = n - m_2 \tag{6}$$

order. If $m_2 < n$, $q = \infty$ is a PP. For $m_2 = n$, $q = \infty$ is a Removable Singularity Point.

For poles and CP $\deg^+(q)$, $q \in \mathcal{P} \cup \mathcal{B} \cup \{\infty\}$, corresponds to the number of outgoing Spectrum Curves at that point. Note that incoming Spectrum Curves alternate with outgoing, so $\deg^+(q) = \deg^-(q)$.

2 Relations between dSLP and graphs properties in the case of integral BC

It is possible to define relations between properties of dSLP and graph theory. Poles or CPs refer to vertices of a certain graph and parts of Spectrum Curves could be



(a) $\boldsymbol{m} = (0, 5)$ (b) $\boldsymbol{m} = (1, 5)$ (c) $\boldsymbol{m} = (2, 5)$ (d) $\boldsymbol{m} = (3, 5)$ (e) $\boldsymbol{m} = (4, 5)$

(f) $\boldsymbol{m} = (0, 4)$ (g) $\boldsymbol{m} = (1, 4)$ (h) $\boldsymbol{m} = (2, 4)$ (i) $\boldsymbol{m} = (3, 4)$ (j) $\boldsymbol{m} = (0, 1)$

(k) $\boldsymbol{m} = (0, 3)$ (l) $\boldsymbol{m} = (1, 3)$ (m) $\boldsymbol{m} = (2, 3)$ (n) $\boldsymbol{m} = (0, 2)$ (o) $\boldsymbol{m} = (1, 2)$

Fig. 5. Spectrum Curves for $n = 6$ [4].

interpreted as edges. In our case, we have a simple balanced weakly connected digraph. Definitions and notations in graphs theory are described in [5].

2.1 Properties of Spectrum Curves

Poles, CPs, regular and nonregular Spectrum Curves, CEPs were found by A. Skučaitė [4].

There is $n-1$ Spectrum Curves for every $n \in \mathbb{N}$, $n \geq 2$. Nonregular Spectrum Curves are CEPs and belong to $\mathbb{R}_x^h = (0, n)$. The number of such Spectrum Curves is equal to

$$\begin{aligned} n_{ce} = &\left\lfloor \frac{n-1}{2n} \mathrm{dbd}(2n; m_+) \right\rfloor + \left\lfloor \frac{n-1}{2n} \mathrm{dbd}(2n; m_-) \right\rfloor \\ &- \left\lfloor \frac{n-1}{2n} \mathrm{dbd}(2n; m_+; m_-) \right\rfloor \end{aligned} \tag{7}$$

Number of regular Spectrum Curves $n_{nce} = n - 1 - n_{ce}$. The poles of CF belong to $\mathbb{R}_x^h \cup \{0\} \cup \{n\} \cup \{\infty\}$ and $n_p + n_\infty = n_{nce}$. So, we have formula

$$n_p + n_{ce} = m_2 - 1. \tag{8}$$

Let us denote

$$\deg_r^+ := \sum_{b\in\mathcal{B}\cap\mathbb{R}_q^h} \deg^+(b), \quad \deg_c^+ := \sum_{b\in\mathcal{B}\cap\mathbb{C}_q^{h\pm}} \deg^+(b) = 2\sum_{b\in\mathcal{B}\cap\mathbb{C}_q^{h+}} \deg^+(b).$$

Let n_c is the number of Spectrum Curves parts in $\mathbb{C}_q^{h+}$ between two CP (including $q=0$ and $q=n$).

2.2 Spectrum domain as a graph

We consider Spectrum domain as graph on sphere (Riemann sphere $\bar{\mathbb{C}}$) because $\mathbb{C}_q^h \cup \{\infty\} \sim S^2$. The poles and CPs of the CF are the vertices of this graph. The point ∞ is the pole or CP.

Lemma 1. *The number of vertices is*

$$\begin{aligned} v &= n_p - n_{2p} + n_b + 1 - \lfloor m_2/n \rfloor \\ &= n_p - n_{2p} + n_{cr} + 2n_{cr}^+ + 1 - \lfloor m_2/n \rfloor \\ &= m_2 - n_{ce} - n_{2p} + n_{cr} + 2n_{cr}^+ - \lfloor m_2/n \rfloor. \end{aligned} \tag{9}$$

We have $e = \sum_{p\in\mathcal{P}} \deg^+(p) + \sum_{b\in\mathcal{B}} \deg^+(b) + \deg^+(\infty)$ ([5]).

Lemma 2. *The number of edges is*

$$e = n_p + \deg_r^+ + \deg_c^+ + n_\infty. \tag{10}$$

Lemma 3. *The number of faces is*

$$\begin{aligned} f &= 2(n_\infty + n_{2p} + n_c - n_{cr}^+ + \lfloor m_2/n \rfloor) \\ &= 2(n - m_2 + n_{2p} + n_c - n_{cr}^+ + \lfloor m_2/n \rfloor). \end{aligned} \tag{11}$$

This lemma is valid for $n_c = n_{cr}^+ = 0$. Each part of spectrum curve between two CPs $b_1, b_2 \in \mathbb{R}_q^h$ increases the number of faces by one. So, this formula is valid for the case $n_{cr}^+ = 0$. Each additional CP $b \in C_q^{h+}$ increases the number of faces by $2(\deg^+(b) - 1)$ and number parts of Spectrum Curves between this CP and other CPs by $2\deg^+(b)$.

Numbers of spectrum vertices, edges and faces, expressed by the formulas above, inserted to the Euler's characteristic's formula of sphere $v - e + f = 2$ give new relation.

Theorem 1. *The Euler's characteristic's formula is equivalent to*

$$\begin{aligned} \sum_{b\in\mathcal{B}} \deg^+(b) = \deg_r^+ + \deg_c^+ &= n_\infty + n_{2p} + 2n_c + n_{cr} - 1 + \lfloor m_2/n \rfloor \\ &= n - m_2 + n_{2p} + 2n_c + n_{cr} - 1 + \lfloor m_2/n \rfloor. \end{aligned} \tag{12}$$

This formula was derived in [4] when there are not CPs in $\mathbb{C}_q^{h\pm}$ ($\deg_c^+ = 0$), all CPs are of the first order ($\deg_r^+ = 2n_{cr}$) and $n_c = 0$. Then it can be rewritten as

$$n_{cr} = n_{2p} + n_\infty - 1 + \lfloor m_2/n \rfloor = n_{2p} + n - m_2 - 1 + \lfloor m_2/n \rfloor.$$

Corollary 1. *The number of edges is*

$$\begin{aligned} e &= 2n_\infty + n_p + n_{2p} + n_{cr} + 2n_c - 1 + \lfloor m_2/n \rfloor \\ &= 2n - m_2 - 2 - n_{ce} + n_{2p} + n_{cr} + 2n_c + \lfloor m_2/n \rfloor \end{aligned} \tag{13}$$

Remark 1. In the case $m_2 < n$ the formulas (9)–(13) are

$$\begin{aligned} v &= m_2 - n_{ce} - n_{2p} + n_{cr} + 2n_{cr}^+, \\ e &= 2n - m_2 - 2 - n_{ce} + n_{2p} + n_{cr} + 2n_c, \\ f &= 2(n - m_2 + n_{2p} + n_c - n_{cr}^+), \\ \deg_r^+ + \deg_c^+ &= n - m_2 - 1 + n_{2p} + 2n_c + n_{cr}, \end{aligned}$$

where n_{ce} is defined by (7).

In the case $m_2 = n$ we have $n_\infty = 0$, $n_p = n - 1 - n_{ce}$ and $n_{2p} = 0$, $n_{cr} = 0$ (so, $\deg_r^+ = 0$) (see [4]). In this case we get $deg_c^+ = 2n_c = 0$ and $n_{cr}^+ = 0$. Thus, for $m_2 = n$ the following formulas

$$\begin{aligned} v &= n - 1 - n_{ce}, \\ e &= n - 1 - n_{ce}, \\ f &= 2 \end{aligned}$$

are valid. Note, that in this case we have Cycle digraph.

Remark 2. If $n_{cr}^+ = 0$ ($\deg_c^+ = 0$) then $\deg_r^+ - n_{cr} = 2n_c + n - m_2 + n_{2p} - 1 > n_{cr}$ shows that there are exist CPs in $\mathbb{R}_q^h$ of the second or the higher order.

References

[1] K. Bingelė, A. Bankauskienė, A. Štikonas. Spectrum curves for a discrete Sturm–Liouville problem with one integral boundary condition. *Nonlinear Anal. Model. Control*, **24**(5):755–774, 2019. https://doi.org/10.15388/NA.2019.5.5.

[2] E. Manstavičius. *Analizinė ir tikimybinė kombinatorika.* TEV, Vilnius, 2007.

[3] M. Sapagovas. *Diferencialinių lygčių kraštiniai uždaviniai su nelokaliosiomis sąlygomis.* Mokslo aidai, Vilnius, 2007.

[4] A. Skučaitė. *Investigation of the spectrum for Sturm–Liouville problem with a nonlocal integral boundary condition.* PhD thesis, Vilnius University, 2016.

[5] J. Vitkauskas, A. Štikonas. Relations between spectrum curves and discrete sturm–liouville problem with nonlocal boundary conditions and graph theory. *Proc. of the Lithuanian Mathematical Society*, **61**:1–6, 2020. https://doi.org/10.15388/LMR.2020.22474.

[6] A. Štikonas, O. Štikonienė. Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions. *Math. Model. Anal.*, **14**(2):229–246, 2009. https://doi.org/10.3846/1392-6292.2009.14.229-246.

REZIUMĖ

Diskrečiojo Šturmo ir Liuvilio uždavinio su nelokaliosiomis kraštinėmis sąlygomis spektrinių kreivių ir grafų teorijos sąsajos. II

J. Vitkauskas, A. Štikonas

Šiame straipsnyje pristatomos sąsajos tarp diskrečiojo Šturmo ir Liuvilio uždavinio su nelokaliąja integraline kraštine sąlyga (poliai, kritiniai taškai ir spektrinės kreivės) bei grafų charakteristikų (viršūnės, briaunos ir veidai). Ankstesnis straipsnis buvo skirtas Sturmo ir Liuvilio uždaviniui su dvitaškėmis nelokaliosiomis kraštinėmis sąlygomis.

Raktiniai žodžiai: Šturmo ir Liuvilio uždavinys; spektrinės kreivės; integralinė kraštinė sąlyga; grafai