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The Mellin transform of the square of the Riemann zeta-function in the critical strip

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Let $s = \sigma + it$ be a complex variable. The Mellin transform $M(f)$ of the function $f(x)$ is defined by

$$M(f) = \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x.$$

However, in analytic number theory usually the modified Mellin transform

$$\int_1^\infty f(x)x^{-s}\,\mathrm{d}x$$

are considered. Let $\zeta(s)$ denote the Riemann zeta-function, and let

$$Z_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} x^{-s}\,\mathrm{d}s, \quad k \geqslant 0,$$

be the modified Mellin transform of $|\zeta(\frac{1}{2} + ix)|^{2k}$. First the function $Z_2(s)$ has been introduced and studied in [6] [7]. The first results for $Z_1(s)$ were obtained in [1].

Define, for $\frac{1}{2} < \rho < 1$,

$$Z_k(\rho, s) = \int_1^\infty \left|\zeta\,(\rho + ix)\right|^{2k} x^{-s}\,\mathrm{d}s, \quad k \geqslant 0.$$

The aim of this note is to obtain an analytic continuation for the function $Z_1(\rho, s)$. Clearly, in view of the estimate

$$\int_1^T \left|\zeta\,(\sigma + ix)\right|^2 \mathrm{d}t \ll_\sigma T, \quad \frac{1}{2} < \sigma < 1,$$

the integral for $Z_1(\rho, s)$ converges absolutely for $\sigma > 1$ and defines there an analytic function. To obtain more precise results, we will apply the asymptotics for the mean square of $\zeta(s)$ in the critical strip.

Let, for $\frac{1}{2} < \sigma < 1$,

$$\int_1^T \left|\zeta\,(\sigma + ix)\right|^2 \mathrm{d}t = \zeta(2\sigma)T + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + E_\sigma(T).$$

Then in [8], see also [2], the estimate

$$E_\sigma(T) = \mathrm{O}\left(T^{\frac{1}{1+4\sigma}} \log^2 T\right) \tag{1}$$

has been proved. Using this, we find, for $\sigma > 1$,

$$\begin{aligned}
Z_1(\rho, s) &= \int_1^\infty x^{-s} d \int_0^x \left|\zeta(\rho + ix)\right|^2 \mathrm{d}x \\
&= x^{-s} \int_1^x \left|\zeta(\rho + ix)\right|^2 \mathrm{d}x \Big|_1^\infty \\
&\quad + s \int_1^\infty \left(\zeta(2\rho)x + (2\pi)^{2\rho-1} \frac{\zeta(2-2\rho)}{2-2\rho} x^{2-2\rho} + E_\rho(x)\right) x^{-s-1} \mathrm{d}x \\
&= s \int_1^\infty \zeta(2\rho) x^{-s} \mathrm{d}x + s(2\pi)^{2\rho-1} \frac{\zeta(2-2\rho)}{2-2\rho} \int_1^\infty x^{1-s-2\rho} \mathrm{d}x \\
&\quad + s \int_1^\infty E_\rho(x) x^{-s-1} \mathrm{d}x \\
&= \frac{s}{1-s} \zeta(2\rho) x^{-s+1} \Big|_1^\infty + s(2\pi)^{2\rho-1} \frac{\zeta(2-2\rho)}{2-2\rho} \frac{x^{2-s-2\rho}}{2-s-2\rho} \Big|_1^\infty \\
&\quad + s \int_1^\infty e_\rho(x) x^{-s-1} \mathrm{d}x \\
&= \frac{s}{s-1} \zeta(2\rho) + \frac{s(2\pi)^{2\rho-1}\zeta(2-2\rho)}{2-2\rho} \frac{1}{2\rho+s-2} + s\int_1^\infty E_\rho(x) x^{-s-1} \mathrm{d}x.
\end{aligned} \tag{2}$$

In view of the estimate (1) the last integral converges absolutely for

$$\frac{1}{1+4\rho} - \sigma - 1 < -1,$$

that is for $\sigma > \frac{1}{1+4\rho}$. Therefore, the equation (2) gives an analytic continuation to the region $\sigma > \frac{1}{1+4\rho}$, except for a simple pole at $s = 1$ with residue $\zeta(2\sigma)$, and a simple pole at $s = 2 - 2\rho$ with residue $\frac{s(2\pi)^{2\rho-1}\zeta(2-2\rho)}{2-2\rho}$. Since $\sigma > \frac{1}{1+4\rho}$ hence we find that $2 - 2\rho > \frac{1}{1-4\rho}$, and $\frac{1}{2} < \rho < \frac{3+\sqrt{17}}{8}$.

Now we will extend the region $\sigma > \frac{1}{1+4\rho}$ of analytic continuation for $Z_1(\rho, s)$. It is known [4], [5], that for $\frac{1}{2} < \sigma < \frac{3}{4}$,

$$\int_2^T E_\sigma^2(t)\,\mathrm{d}t = \frac{2}{5-4\sigma} (2\pi)^{2\sigma-\frac{3}{2}} \frac{\zeta^2\left(\frac{3}{2}\right)}{\zeta(3)} \zeta\left(\frac{5}{2} - 2\sigma\right) \zeta\left(\frac{1}{2} + 2\sigma\right) T^{\frac{5}{2}-2\sigma} + \mathrm{O}(T),$$

$$\int_2^T E_{\frac{3}{4}}^2(t)\,\mathrm{d}t = \frac{\zeta^2\left(\frac{3}{2}\right)\zeta(2)}{\zeta(3)} T \log T + \mathrm{O}\left(T \log^{\frac{1}{2}} T\right),$$

and, for $\frac{3}{4} < \sigma < 1$,

$$\int_2^T E_\sigma^2(t)\,\mathrm{d}t \ll T.$$

Therefore, we have, for $0 < \alpha < \sigma + \frac{1}{2}$,

$$\begin{aligned}
&\int_1^\infty E_\rho(x)x^{-\sigma-1}\,\mathrm{d}x \\
&\ll \left(\int_1^\infty E_\rho^2(x)x^{-2\alpha}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_1^\infty x^{2\alpha-2\sigma-2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\ll \left(\int_1^\infty E_\rho^2(x)x^{-2\alpha}\,\mathrm{d}x\right)^{\frac{1}{2}} \ll \left(\int_1^\infty x^{-2\alpha}\,\mathrm{d}\int_1^x E_\rho^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&= x^{-2\alpha}\int_1^x E_\rho^2(x)\,\mathrm{d}x\Big|_1^\infty + 2\alpha\int_1^\infty x^{\frac{5}{2}-2\rho}x^{-2\alpha-1}\,\mathrm{d}x \\
&\ll x^{\frac{5}{2}-2\alpha-2\rho}\Big|_1^\infty + \int_1^\infty x^{\frac{3}{2}-2\alpha-2\rho}\,\mathrm{d}x. \qquad (3)
\end{aligned}$$

Let $2\alpha > \frac{5}{2} - 2\rho$. Then in view of (3), since then $\frac{3}{2} - 2\rho - 2\alpha < -1$, we find that

$$\int_1^\infty E_\rho(x)x^{-\sigma-1}\,\mathrm{d}x \ll 1.$$

Since $0 < \alpha < \sigma + \frac{1}{2}$, here $\sigma > \frac{3}{4} - \rho$. Then the integral

$$\int_1^\infty E_\rho(x)x^{-s-1}\,\mathrm{d}x$$

converges uniformly on compact subsets of the half-plane $\sigma > \frac{3}{4} - \rho$. Hence equality (2) gives analytic continuation for the function $Z_1(\rho, s)$ to the region $\sigma > \frac{3}{4} - \rho$, except for the simple poles at $s = 1$ and $s = 2 - 2\rho$.

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REZIUMĖ

M. Stoncelis. Rymano dzeta funkcijos kvadrato Melino tranformacijos kritinėje juostoje

Gautas Rymano dzeta funkcijos kvadrato Melino transformacijos kritinėje juostoje analizinis pratęsimas į kairę nuo vienetinės tiesės.