Liet. matem. rink, **46***, spec. nr.*, 2006, 454–458

On Runge–Kutta-type methods for solving multidimensional stochastic differential equations

Jurgis NAVIKAS (VU)
e-mail: jnavikas@hotmail.com

We consider a six-dimensional Stratonovich stochastic differential equation of the form

$$X_i(t) = x_i + \int_0^t f_i(X(t))\,\mathrm{d}t + \sum_{j=1}^{6} \int_0^t g_{ij}(X(t)) \circ \mathrm{d}B_j(t), \quad t \in [0,T],\ i = 1,\dots,6,$$

or, in matrix notation,

$$X(t) = x + \int_0^t f(X(t))\,\mathrm{d}t + \int_0^t g(X(t))\,\mathrm{d}B(t), \quad t \in [0,T], \tag{1}$$

where B is a six-dimensional Brownian motion, and $f_i\colon \mathbb{R}^6 \to \mathbb{R}$, $g_{ij}\colon \mathbb{R}^6 \to \mathbb{R}$, $i, j = 1, 2, \dots, 6$.

A family of stochastic processes $\{X^h,\ h > 0\}$ is called a second-order weak approximation of the solution X (in the time interval $[0,T]$) if, for all $t \in [0,T]$,

$$E\varphi(X^h(t)) - E\varphi(X(t)) = \mathrm{O}(h^2), \quad h \to 0.$$

for a 'rather wide' class of functions $\varphi\colon \mathbb{R}^6 \to \mathbb{R}$ (see [1], [2]).

Similarly to [2], [4], [5], we consider Runge–Kutta-type approximations

$$X^h(0) = x, \quad X^h(t_{k+1}) = A\big(X^h(t_k), h, \Delta B_k\big), \quad k = 0, 1, \dots, N,$$

where $t_k = t_k(h) = kh$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and the function $A\colon \mathbb{R}^6 \times [0,T] \times \mathbb{R}^6 \to \mathbb{R}^6$ is of the form

$$A(x,s,b) := x + \sum_{i=0}^{3} q_i F_i s + \sum_{i=0}^{3} r_i G_i b, \tag{2}$$

where

$$F_0 = f(x), \quad G_0 = g(x + \alpha_{00} F_0 s),$$
$$F_1 = f(x + \alpha_{10} F_0 s + \beta_{10} G_0 b), \quad G_1 = g(x + \alpha_{10} F_0 s + \beta_{10} G_0 b),$$
$$F_2 = f\big(x + (\alpha_{20} F_0 + \alpha_{21} F_1)s + (\beta_{20} G_0 + \beta_{21} G_1)b\big),$$

$$G_2 = g\big(x + (\alpha_{20}F_0 + \alpha_{21}F_1)s + (\beta_{20}G_0 + \beta_{21}G_1)b\big),$$
$$F_3 = f\big(x + (\alpha_{30}F_0 + \alpha_{31}F_1 + \alpha_{32}F_2)s + (\beta_{30}G_0 + \beta_{31}G_1 + \beta_{32}G_2)b\big),$$
$$G_3 = g\big(x + (\alpha_{30}F_0 + \alpha_{31}F_1 + \alpha_{32}F_2)s + (\beta_{30}G_0 + \beta_{31}G_1 + \beta_{32}G_2)b\big). \quad (3)$$

In [6], we derived equations for the parameters of the approximation of order two (in the weak sense) and found 'nice' examples of the coefficients for a two-dimensional SDE with the diffusion matrix of the form

$$g(x_1, x_2) = \begin{pmatrix} C & 0 \\ 0 & g(x_1, x_2) \end{pmatrix}. \quad (4)$$

In the multidimensional case, such a derivation seems to be rather difficult. Therefore, here we try another approach. We take the parameters of the two-dimensional and directly check that they are suitable in the six-dimensional analogue of the diffusion matrix (4). Precisely, we consider six-dimensional SDEs with diffusion matrices of the form

$$g(x) = \begin{pmatrix} c_1 & c_2 & c_3 & 0 & 0 & 0 \\ c_4 & c_5 & c_6 & 0 & 0 & 0 \\ c_7 & c_8 & c_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1(x_1, x_2, x_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & g_2(x_1, x_2, x_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & g_3(x_1, x_2, x_3) \end{pmatrix}. \quad (5)$$

In the two-dimensional case, we have obtained the following values for the parameters q_i, r_i, α_{ij}, β_{ij} written in the Butcher-type array [6]:

$$\begin{array}{c|cccc} \alpha_0 & \alpha_{00} & & & \\ \alpha_1 & \alpha_{10} & & & \\ \alpha_2 & \alpha_{20} & \alpha_{21} & & \\ \alpha_3 & \alpha_{30} & \alpha_{31} & \alpha_{32} & \\ \hline & q_0 & q_1 & q_2 & q_3 \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \beta_1 & \beta_{10} & & & \\ \beta_2 & \beta_{20} & \beta_{21} & & \\ \beta_3 & \beta_{30} & \beta_{31} & \beta_{32} & \\ \hline & r_0 & r_1 & r_2 & r_3 \end{array}$$

$$\begin{array}{c|cccc} 1 & 1 & & & \\ 0 & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & \\ 1 & 1 & 0 & 0 & \\ \hline & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

We verify that the sufficient conditions [2], [3] for the second-order accuracy of a weak approximation are satisfied by the corresponding Runge–Kutta approximation. Using MAPLE, we first check conditions $(F_1A)_i$, $(F_2A)_{ij}$, $i, j = 1, \dots, 6$.

On the left-hand and right-hand sides of conditions $(F_1A)_i$, we respectively have

$$\left(\frac{\partial}{\partial s}A_i + \frac{1}{2}\sum_p \frac{\partial^2}{\partial b_p^2}A_i\right)(\bar{x}) = f_i(x_1, x_2, x_3, x_4, x_5, x_6)$$

and

$$\left(f_i + \frac{1}{2}\sum_{l,j} g_{lj}\frac{\partial g_{ij}}{\partial x_l}\right)(x) = f_i(x_1, x_2, x_3, x_4, x_5, x_6).$$

Next, we consider conditions $(F_2A)_{ij}$. For example, on the left-hanf side of $(F_2A)_{11}(x)$, we have

$$\sum_p \frac{\partial}{\partial b_p}A_1 \frac{\partial}{\partial b_p}A_1(\bar{x}) = c_1^2 + c_2^2 + c_3^2,$$

and, on the right-hand side, we have

$$a_{11}(x) = c_1^2 + c_2^2 + c_3^2.$$

Further, on the left-hand side of $(F_2A)_{44}(x)$, we have

$$\sum_p \frac{\partial}{\partial b_p}A_1 \frac{\partial}{\partial b_p}A_1(\bar{x}) = g_1^2(x_1, x_2, x_3),$$

and, on the right-hand side, we have

$$a_{44}(x) = g_1^2(x_1, x_2, x_3).$$

Similarly, one can check the remaining conditions $(F_2A)_{ij}(x)$, $i, j = 1, \dots, 6$. Thus, our Runge–Kutta approximation satisfies the first-order accuracy conditions.

Now we consider conditions $(S_1A)_i(x)$. For example, on the left-hand side of $(S_1A)_1(x)$, we have

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial s^2}A_1 + \sum_p \frac{\partial^3}{\partial b_p^2 \partial s}A_1 + \frac{1}{4}\sum_{p,q}\frac{\partial^4}{\partial b_p^2 \partial b_q^2}A_1\right)(\bar{x}) \\
&= f_1(x_1, x_2, x_3, x_4, x_5, x_6)\frac{\partial}{\partial x_1}f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
&\quad + f_2(x_1, x_2, x_3, x_4, x_5, x_6)\frac{\partial}{\partial x_2}f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
&\quad + f_3(x_1, x_2, x_3, x_4, x_5, x_6)\frac{\partial}{\partial x_3}f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
&\quad + f_4(x_1, x_2, x_3, x_4, x_5, x_6)\frac{\partial}{\partial x_4}f_1(x_1, x_2, x_3, x_4, x_5, x_6)
\end{aligned}$$

$$
\begin{aligned}
&+ f_5(x_1, x_2, x_3, x_4, x_5, x_6)\frac{\partial}{\partial x_5} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ f_6(x_1, x_2, x_3, x_4, x_5, x_6)\frac{\partial}{\partial x_6} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ 1/2(c_1^2 + c_2^2 + c_3^2)\frac{\partial^2}{\partial x_1^2} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ (c_1c_4 + c_2c_5 + c_3c_6)\frac{\partial^2}{\partial x_1 \partial x_2} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ (c_1c_7 + c_2c_8 + c_3c_9)\frac{\partial^2}{\partial x_1 \partial x_3} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ 1/2(c_4^2 + c_5^2 + c_6^2)\frac{\partial^2}{\partial x_2^2} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ (c_4c_7 + c_5c_8 + c_6c_9)\frac{\partial^2}{\partial x_2 \partial x_3} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ 1/2(c_7^2 + c_8^2 + c_9^2)\frac{\partial^2}{\partial x_3^2} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ 1/2 g_1(x_1, x_2, x_3)^2\frac{\partial^2}{\partial x_4^2} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ 1/2 g_2(x_1, x_2, x_3)^2\frac{\partial^2}{\partial x_5^2} f_1(x_1, x_2, x_3, x_4, x_5, x_6)\\
&+ 1/2 g_3(x_1, x_2, x_3)^2\frac{\partial^2}{\partial x_6^2} f_1(x_1, x_2, x_3, x_4, x_5, x_6),
\end{aligned}
$$

and we have the same expression on the right-hand side.

Similarly, we check the remaining conditions $(S_1A)_i(x)$, $(S_2A)_{ij}(x)$, and $(S_3A)_{ijk}(x)$ for the second-order accuracy.

Thus, we have constructed a second-order Runge–Kutta approximation for the case (5).

Applications. In physics, the stochastic particle equations of motion that follow from the Fokker–Planck equation are

$$
\begin{aligned}
r' &= v,\\
v' &= \frac{\mathbf{F}}{m} - \nu v + \sqrt{D}\Gamma(t),
\end{aligned}
$$

where **F** is the force including both the external force and the self-generated mean field space charge force, m is the mass of particle, ν is the friction coefficient, D is the diffusion coefficient, and $\Gamma(t)$ is Gaussian white noise.

In the case of three dimensions, the dynamical equations then take the general form

$$\begin{aligned}
\dot{x_1} &= F_1(x_1, x_2, x_3, x_4, x_5, x_6) + \sigma_{11}(x_2, x_4, x_6)\xi_1(t),\\
\dot{x_2} &= F_2(x_1),\\
\dot{x_3} &= F_3(x_1, x_2, x_3, x_4, x_5, x_6) + \sigma_{33}(x_2, x_4, x_6)\xi_3(t),\\
\dot{x_4} &= F_4(x_3),\\
\dot{x_5} &= F_5(x_1, x_2, x_3, x_4, x_5, x_6) + \sigma_{55}(x_2, x_4, x_6)\xi_5(t),\\
\dot{x_6} &= F_6(x_5).
\end{aligned} \tag{6}$$

In Eqs. 6, the indices are single-particle phase-space coordinate indices; the convention used here is that the odd indices correspond to momenta, and the even indices to the spatial coordinate. In the dynamical equations for the momenta, the first term on the right-hand side is a systematic drift term which includes the effects due to external forces and damping. The second term is stochastic in nature and describes a noise force which, in general, is a function of position.

For solution of such dynamical equations, the constructed Runge–Kutta approximation is an effective numerical method.

References

1. P.E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, Berlin, Springer-Verlag (1980).
2. V. Mackevičius, Second order weak approximations for Stratonovich stochastic differential equations, *Lith. Math. J.*, **34**(2), 183–200 (1994).
3. V. Mackevičius, Extrapolation of approximations of solutions of stochastic differential equations, in: *Proc. Seventh Japan–Russia Symp. Prob. Th. Math. Stat.*, Tokyo, 1995, World Sc. Publ. Corp., Singapore (1996), pp. 276–297.
4. V. Mackevičius, J. Navikas, Second order weak Runge–Kutta type methods for Itô equations, *Math. Comp. Simulation*, **57**(1–2), 29–34 (2001).
5. G.B. Loghmani, High strong order implicit Runge–Kutta methods for stochastic ordinary differential equations, in: *22nd System Dynamics Conference* (2004).
6. J. Navikas, On Runge–Kutta type methods for two dimensional stochastic differential equations, *Lith. Math. J.*, **46**(3), 403–412 (2006).
7. D. Talay, L. Tubaro, Expansion of the global error for numerical schemes solving stochastic differential equations, *Stoch. Anal. Appl.*, **8**(4), 94–120 (1990).
8. G.N. Milstein, Weak approximation of the solution of systems of stochastic differential equations, *Theor. Prob. Appl.*, **30**, 750–766 (1985).

REZIUMĖ

J. Navikas. Stochastinių diferencialinių lygčių sprendimo Rungės–Kuto metodai

Daugiamačiu atveju antrosios eilės silpnosioms Rungės–Kuto aproksimacijas stochastinėms diferencialinėms lygtims (SDL) konstruoti yra gana sudėtinga. Šiame straipsnelyje pabandyta apeiti šiuos sunkumus tokiu būdu: įrodyta, kad šešiamatės SDL aproksimacijai tinka dvimatės SDL Rungės–Kuto aproksimacijos koeficientai.