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Netolygūs įverčiai puasoninėje aproksimacijoje

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Tegul $X_{n1}, X_{n2}, \dots, X_{nk_n}$, $k_n \to \infty$, kai $n \to \infty$, – nepriklausomų atsitiktinių dydžių serijų seka. Atsitiktinio dydžio X_{nj}, $j = 1, 2, \dots, k_n$, charakteristinę funkciją

$$f_j(t) = f_{nj}(t) = \mathrm{E}\mathrm{e}^{\mathrm{i}tX_{nj}},$$

jei $\mathrm{E}|X_{nj}|^s < \infty$, taško $z = z(\mathrm{i}t) = \mathrm{e}^{\mathrm{i}t} - 1$ aplinkoje, parašysime pavidalu

$$f_j(t) = 1 + \sum_{k=1}^{s} \frac{\alpha_{jk}}{k!}(\mathrm{e}^{\mathrm{i}t} - 1)^k + \mathrm{o}(|\mathrm{e}^{\mathrm{i}t} - 1|^s), \tag{1}$$

Čia α_{jk} – atsitiktinio dydžio X_{nj} faktorialiniai momentai, t.y.

$$\alpha_{jk} = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\mathrm{E}(1+z)^{X_{nj}}\Big|_{z=0} = \mathrm{E}X_{nj}(X_{nj} - 1)\cdots(X_{nj} - k + 1),$$

o faktorialiniai kumuliantai

$$\gamma_{jk} = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\log \mathrm{E}(1+z)^{X_{nj}}\Big|_{z=0}, \quad k = 1, 2, \dots .$$

Šiame darbe nagrinėjami nepriklausomų atsitiktinių dydžių, įgyjančių sveikąsias neneigiamas reikšmes, sumų $S_{nk_n} = X_{n1} + X_{n2} + \dots + X_{nk_n}$ skirstinių aproksimacijos Puasono dėsniu ([1], [2], [3]) liekamojo nario netolygūs įverčiai. Puasono skirstinio su parametru λ pasiskirstymo funkciją žymėsime

$$\Pi(x;\lambda) = \sum_{l=0}^{[x]} \pi(l;\lambda) = \sum_{l=0}^{[x]} \frac{\lambda^l}{l!}\mathrm{e}^{-\lambda}.$$

Be to, žymėsime

$$\pi_1(l;\lambda) = \pi(l;\lambda) - \pi(l-1;\lambda), \quad \pi(l;\lambda) = 0, \ \ l = -1, -2, \dots$$

Visi rezultatai gauti, pritaikius lemą, kurią tik suformuluosime.

LEMA. *Jei atsitiktinių dydžių X ir Y, įgyjančių sveikąsias reikšmes, pasiskirstymo funkcijos $F(x)$ ir $G(x)$, charakteristinės funkcijos $f(t)$ ir $g(t)$, o $\Delta(x) = F(x) - G(x)$*

ir $\delta(t) = f(t) - g(t)$, *tai*

$$(1+|x|)|\Delta(x)| \leqslant \frac{1}{8\pi}\left\{4\int_{-\pi}^{\pi}\left|\frac{\delta(t)}{\sin\frac{t}{2}}\right|\mathrm{d}t + \int_{-\pi}^{\pi}\left|\frac{\delta(t)}{\sin^2\frac{t}{2}}\right|\mathrm{d}t + 2\int_{-\pi}^{\pi}\left|\frac{\delta'(t)}{\sin\frac{t}{2}}\right|\mathrm{d}t\right\} \tag{2}$$

ir

$$(1+|x|)^2|\Delta(x)| \leqslant \frac{1}{8\pi}\left\{8\int_{-\pi}^{\pi}\left|\frac{\delta(t)}{\sin\frac{t}{2}}\right|\mathrm{d}t + 5\int_{-\pi}^{\pi}\left|\frac{\delta(t)}{\sin^2\frac{t}{2}}\right|\mathrm{d}t + \int_{-\pi}^{\pi}\left|\frac{\delta(t)}{\sin^3\frac{t}{2}}\right|\mathrm{d}t\right.$$
$$\left. + 8\int_{-\pi}^{\pi}\left|\frac{\delta'(t)}{\sin\frac{t}{2}}\right|\mathrm{d}t + 2\int_{-\pi}^{\pi}\left|\frac{\delta'(t)}{\sin^2\frac{t}{2}}\right|\mathrm{d}t + 2\int_{-\pi}^{\pi}\left|\frac{\delta''(t)}{\sin\frac{t}{2}}\right|\mathrm{d}t\right\}. \tag{3}$$

1 TEOREMA. *Jeigu nepriklausomi kiekvienoje serijoje atsitiktiniai dydžiai* X_{nj}, $j = 1, 2, \ldots, k_n$, $n = 1, 2, \ldots$ *įgyja neneigiamas sveikąsias reikšmes, turi vidurkius* $\mathrm{E}X_{nj} = \lambda_j^{(n)} > 0$, *ir*

$$\overline{\alpha}_{jk} = \mathrm{E}\left|X_{nj}(X_{nj}-1)\cdots(X_{nj}-k+1)\right| \leqslant \frac{k!\lambda_j^{(n)}}{\Delta_n^{k-1}}, \quad k = 2, 3, \tag{4}$$

o

$$10 \leqslant \Delta_n \leqslant \Big(\max_{1\leqslant j\leqslant k_n}\lambda_j^{(n)}\Big)^{-1}, \tag{5}$$

tai

$$(1+|x|)\Big|\mathrm{P}\{S_{nk_n} < x\} - \Pi(x;\lambda) - \frac{1}{2}\Gamma_2\pi_1\big([x];\lambda\big)\Big| \leqslant 49\frac{\lambda}{\Delta_n^2},$$

čia $\lambda = \sum_{j=1}^{k_n}\lambda_j^{(n)}$, $\Gamma_2 = \sum_{j=1}^{k_n}\gamma_{j2}$.

Įrodymas. Įvertinsime

$$|\delta_n(t)| = \left|f_{S_{nk_n}}(t) - g(t)\right| = \left|f_{S_{nk_n}}(t) - \mathrm{e}^{\lambda(\mathrm{e}^{\mathrm{i}t}-1)}\Big(1 + \frac{1}{2}\Gamma_2(\mathrm{e}^{\mathrm{i}t}-1)\Big)^2\right|.$$

Analogiškai, kaip ir [2], gauname

$$f_j(t) = 1 + \alpha_{j1}(\mathrm{e}^{\mathrm{i}t}-1) + \frac{\alpha_{j2}}{2!}(\mathrm{e}^{\mathrm{i}t}-1)^2 + \frac{\beta_{j3}}{2!}(\mathrm{e}^{\mathrm{i}t}-1)^3, \tag{6}$$

čia

$$\beta_{j3} = \int_0^1 (1-\tau)\varphi_{nj}'''\big(\tau(\mathrm{e}^{\mathrm{i}t}-1)\big)\mathrm{d}\tau,$$

o $\varphi_{nj}(z) = \mathrm{E}(1+z)^{X_{nj}}$. Kai $z_0 = \tau(\mathrm{e}^{\mathrm{i}t}-1)$, tai

$$|\varphi_{nj}'''(z_0)| = \left|\mathrm{E}X_{nj}(X_{nj}-1)(X_{nj}-2)(1+z_0)^{X_{nj}-3}\right| \leqslant \overline{\alpha}_{j3},$$

ir

$$|\beta_{j3}| \leqslant \frac{1}{3}\overline{\alpha}_{j3}.$$

Tada

$$\exp\log f_j(t) = \exp\Big\{\alpha_{j1}(\mathrm{e}^{\mathrm{i}t}-1) + \frac{\gamma_{j2}}{2!}(\mathrm{e}^{\mathrm{i}t}-1)^2 + r_{j1}(t) + r_{j2}(t)\Big\},$$

čia

$$r_{j1}(t) = \Big(\frac{\beta_{j3}}{2!} - \alpha_{j1}\frac{\alpha_{j2}}{2!}\Big)(\mathrm{e}^{\mathrm{i}t}-1)^3 - \frac{1}{2}\bigg(\Big(\frac{\alpha_{j2}}{2!}\Big)^2 + \alpha_{j1}\beta_{j3}\bigg)(\mathrm{e}^{\mathrm{i}t}-1)^4$$
$$- \frac{\alpha_{j2}\beta_{j3}}{2!2!}(\mathrm{e}^{\mathrm{i}t}-1)^5 - \frac{1}{2}\frac{\beta_{j3}^2}{(2!)^2}(\mathrm{e}^{\mathrm{i}t}-1)^6$$

ir

$$r_{j2}(t) = \sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\Big(\alpha_{j1}(\mathrm{e}^{\mathrm{i}t}-1) + \frac{\alpha_{j2}}{2!}(\mathrm{e}^{\mathrm{i}t}-1)^2 + \frac{\beta_{j3}}{2!}(\mathrm{e}^{\mathrm{i}t}-1)^3\Big)^k.$$

Tegul

$$r(t) = \sum_{j=1}^{k_n}\big(r_{j1}(t) + r_{j2}(t)\big).$$

Tada

$$\exp\log f_{S_{nk_n}}(t) = \exp\Big\{\sum_{j=1}^{k_n}\log f_j(t)\Big\}$$
$$= \exp\Big\{\lambda(\mathrm{e}^{\mathrm{i}t}-1) + \frac{\Gamma_2}{2!}(\mathrm{e}^{\mathrm{i}t}-1)^2 + r(t)\Big\}. \tag{7}$$

Iš (4), (5), gauname

$$|r(t)| \leqslant 3,06\frac{\lambda}{\Delta_n^2}|\mathrm{e}^{\mathrm{i}t}-1|^3.$$

Iš (4), turime

$$\frac{1}{2}|\Gamma_2| \leqslant \frac{\lambda}{\Delta_n}.$$

Kadangi $|\mathrm{e}^{\lambda(\mathrm{e}^{\mathrm{i}t}-1)}| = \mathrm{e}^{-2\lambda\sin^2\frac{t}{2}}$ ir $|\mathrm{e}^{it}-1| = 2|\sin\frac{t}{2}|$, tai

$$|\delta_n(t)| = \big|f_{S_{nk_n}}(t) - g(t)\big| \leqslant 32,28\frac{\lambda}{\Delta_n^2}\Big|\sin\frac{t}{2}\Big|^3\mathrm{e}^{-\lambda\sin^2\frac{t}{2}}. \tag{8}$$

Analogiškai įvertiname ir $|\delta_n'(t)| = |f'_{S_{nk_n}}(t) - g'(t)|$. Mūsų atveju,

$$|\delta_n'(t)| = \left|f'_{S_{nk_n}}(t) - g'(t)\right| \leqslant 72,55\frac{\lambda}{\Delta_n^2}\sin^2\frac{t}{2}\,\mathrm{e}^{-\lambda\sin^2\frac{t}{2}}. \tag{9}$$

Įstatę (8) ir (9) į (2), gauname

$$(1+|x|)\Big|\mathrm{P}\{S_{nk_n} < x\} - \Pi(x;\lambda) - \frac{1}{2}\Gamma_2\,\pi_1([x];\lambda)\Big| \leqslant 49\frac{\lambda}{\Delta_n^2}.$$

Analogiškai įrodoma kita teorema.

2 TEOREMA. *Jeigu nepriklausomi kiekvienoje serijoje atsitiktiniai dydžiai* X_{nj}, $j = 1, 2, \ldots, k_n$, $n = 1, 2, \ldots$ *įgyja neneigiamas sveikąsias reikšmes, turi vidurkius* $\mathrm{E}X_{nj} = \lambda_j^{(n)} > 0$, *ir*

$$\overline{\alpha}_{jk} = \mathrm{E}\big|X_{nj}(X_{nj}-1)\cdots(X_{nj}-k+1)\big| \leqslant \frac{k!\lambda_j^{(n)}}{\Delta_n^{k-1}}, \quad k = 2, 3, \tag{10}$$

o

$$10 \leqslant \Delta_n \leqslant \Big(\max_{1\leqslant j\leqslant k_n}\lambda_j^{(n)}\Big)^{-1}, \tag{11}$$

tai

$$(1+|x|)^2\Big|\mathrm{P}\{S_{nk_n} < x\} - \Pi(x;\lambda) - \frac{1}{2}\Gamma_2\,\pi_1([x];\lambda)\Big| \leqslant C\frac{\lambda}{\Delta_n^2},$$

čia $\lambda = \sum_{j=1}^{k_n}\lambda_j^{(n)}$, $\Gamma_2 = \sum_{j=1}^{k_n}\gamma_{j2}$, *o* C *– absoliuti konstanta.*

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SUMMARY

K. Padvelskis. Nonuniform estimates in the approximation by the Poisson law

Poisson approximation for the sum of independent random variables is investigates in this paper.

Keywords: Poisson distribution, independent random variables, nonuniform estimates.