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On the mean value of coefficients of certain cusp forms

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Let $F(z)$ be a normalized eigenform of weight κ with the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi i m z}, \quad c(1) = 1.$$

The zeta-function $\varphi(s, F)$, $s = \sigma + it$, attached to $F(z)$, for $\sigma > \frac{\kappa+1}{2}$, is defined by

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and it is analytically continuable to an entire function.

In [1] the mean value

$$\sum_{m \leqslant x} h(m)$$

was considered. Here $h(m) = g_w^2(m)m^{1-\kappa}$, where $g_w(m)$ is defined by

$$\varphi^w(s, w) = \sum_{m=1}^{\infty} \frac{g_w(m)}{m^s}, \quad \sigma > \frac{\kappa+1}{2}.$$

Note that $g_w(m)$ is a multiplicative function,

$$g_w(p^k) = \sum_{l=0}^{k} d_w(p^l)\alpha^l(p)d_w(p^{k-l})\beta^{k-l}(p), \tag{1}$$

where

$$d_w(p^k) = \frac{w(w+1)\dots(w+k-1)}{k!}, \quad k = 1, 2, \dots,$$

and

$$c(p) = \alpha(p) + \beta(p).$$

The aim of this note is to give an asymptotic formula for the mean value of $h(m)$ with explicitly given remainder term.

Let

$$m(h,x)=\prod_{p\leqslant x}\left(1+\sum_{\alpha=1}^{\infty}\frac{h(p^\alpha)}{p^\alpha}\right),$$

$\Gamma(s)$ denote the Euler gamma-function, and let γ be the Euler constant.

Let $c_p=c(p)p^{\frac{1-\kappa}{2}}$. Then by (1) $g_w(p)=w(\alpha(p)+\beta(p))=wc(p)$. Hence $h(p)=w^2c_p^2$. In [2] it was proved that, for $x\to\infty$,

$$\sum_{p\leqslant x}c_p^2\log p=x(1+\mathrm{o}(1)).$$

Consequently, we have that there exists a function $r_1(x)$, $r_1(x)\to 0$ as $x\to\infty$, such that

$$\sum_{p\leqslant x}h(p)\log p=w^2x+w^2xr_1(x). \tag{2}$$

Let $r_2(x)=\sup\limits_{z\geqslant x}|r_1(z)|$. Then in view of (2)

$$\sup_{z\geqslant x}\left|\frac{1}{z}\sum_{p\leqslant z}h(p)\log p-w^2\right|=w^2r_2(x).$$

Define

$$r_3(x)=\max\bigl(w^2r_2(x),(\log x)^{-1}\bigr),$$

and put

$$r(x)=\max\bigl(w^{-2}r_3(x),(\log x)^{-1}\bigr).$$

Then, clearly, $r(x)\to 0$ as $x\to\infty$, and $r(x)\geqslant r_2(x)$.

Suppose that $y=x^{\frac{1}{\sqrt{\log\log x}}}$. Let

$$S_1(x,y)=\sum_{m_1\leqslant x}h(m_1),$$

and $\epsilon(x)=r(y)$.

THEOREM. *Let* $\frac{c}{\sqrt{\log\log x}}\leqslant|w|\leqslant\frac{1}{2}$ *and* $\mathrm{Re}\,w^2>0$. *Then uniformly in* w *and* x

$$\sum_{m\leqslant x}h(m)=\frac{\mathrm{e}^{-\gamma w^2}x}{\Gamma(w^2)\log x}m(h,x)$$

$$\times\left(\sqrt{\log\log x}\right)^{\mathrm{O}(|w|^2\varepsilon(x))}\left(1+\mathrm{O}\left(|w|^2\exp\left\{-\frac{\log x}{\sqrt{\log\log x}}\right\}\right)\right)$$

$$+\mathrm{O}\Big(\frac{|w|^2 x m(|h|,x)}{\log x}\big(\sqrt{\log\log x}\big)^{-1+\mathrm{O}(|w|^2\varepsilon(x))}\Big)$$
$$+\mathrm{O}\Big(\frac{|w|^2\varepsilon(x)x}{\log x}m\big(|h|,x\big)\big(\sqrt{\log\log x}\big)^{|w|^2}\Big).$$

The theorem is based on the following assertion.

LEMMA 1. *Let* $\frac{c}{\sqrt{\log\log x}} \leqslant |w| \leqslant \frac{1}{2}$ *and* $\operatorname{Re} w^2 > 0$. *Then uniformly in w and x*

$$S_1(x,y) = 1 + \frac{x\rho_w'(\sqrt{\log\log x})}{\log y} - \frac{w^2 y}{\log y}$$
$$+\mathrm{O}\Big(|w|^2 x\frac{r(y)}{\log y}\big(\sqrt{\log\log x}\big)^{|w|^2-1}\frac{m(|h|,x)}{m(|h|,y)}$$
$$+\frac{x r(y)\log y}{y}\big(\sqrt{\log\log x}\big)^{|w|^2-1}\Big).$$

Here $\rho_w(u)$, *for* $u \geqslant 0$, *is a continuous solution of the difference – differential equation*

$$u\rho_w'(u) = w^2\rho_w(u-1),$$

and $\rho_w(u) = 0$ *for* $u < 0$.

Proof. Let

$$Z(x,y) = \sum_{m_1\leqslant x} h(m_1)\log m_1.$$

Then, obviously,

$$Z(x,y) = S_1(x,y)\log x - \int_1^x S_1(u,y)\frac{\mathrm{d}u}{u}. \tag{3}$$

Moreover, by (2) and Lemma 5 of [1]

$$Z(x,y) = \sum_{\substack{p^\alpha\leqslant xp\\ p>y}} h(p^\alpha)\log p^\alpha \sum_{\substack{m_1\leqslant \frac{x}{p^\alpha}\\ (m_1,p)=1}} h(m_1)$$
$$= \sum_{m_1\leqslant\frac{x}{y}} h(m_1) \sum_{\substack{p^\alpha\leqslant\frac{x}{m_1}\\ p>y}} h(p^\alpha)\log p^\alpha$$
$$= \sum_{m_1\leqslant\frac{x}{y^2}} h(m_1) \sum_{\substack{p^{\alpha+\beta}\leqslant\frac{x}{m_1}\\ p>y,\, p\nmid m_1}} h(p^\alpha)h(p^\beta)\log p^\alpha$$

$$= \sum_{m_1 \leqslant \frac{x}{y^2}} h(m_1)\Big(w^2\Big(\frac{x}{m_1} - 1\Big) + \mathrm{O}\Big(|w|^2\Big(r(y)\frac{x}{m_1} + \sqrt{\frac{x}{m_1}}\Big)\Big)\Big)$$

$$+\mathrm{O}\Big(|w|^4 \frac{x}{y} \sum_{m_1 \leqslant \frac{x}{y^2}} \frac{|h(m_1)|}{m_1}\Big). \tag{4}$$

This and (3) show that, for $x = y^t$,

$$tS_1(y^t, y) - \int_0^t S_1(y^u, y)\,\mathrm{d}u = \mathrm{O}\Big(|w|^2 r(y)\frac{y^t}{\log y} \sum_{m_1 \leqslant y^t} \frac{|h(m_1)|}{m_1}\Big)$$

$$= \mathrm{O}\Big(|w|^2 r(y)\frac{y^t}{\log y}\frac{m(|h|, y^t)}{m(|h|, y)}\Big).$$

Therefore

$$\frac{1}{t}\big(S_1(y^t, y) - 1\big) - \frac{1}{t^2}\int_1^t (S_1(y^u, y) - 1)\,\mathrm{d}u = \mathrm{O}\Big(|w|^2 r(y)\frac{y^t}{t^2 \log y}\frac{m(|h|, y^t)}{m(|h|, y)}\Big),$$

and

$$\frac{1}{t}\int_1^t \big(S_1(y^u, y) - 1\big)\,\mathrm{d}u = \mathrm{O}\Big(|w|^2 r(y)\frac{y^t}{t^2 \log^2 y}\frac{m(|h|, y^t)}{m(|h|, y)}\Big). \tag{5}$$

Since

$$t = \frac{\log x}{\log y} = \frac{w^2 r_3(y) y^t \log y^t}{w^2 r_3(y) y^t \log y} = \mathrm{O}\Big(\frac{|w|^2 r(y) y^t}{\log y}\Big),$$

(3)–(5) imply

$$tS_1(y^t, y) = \frac{w^2 y^t}{\log y}\sum_{m_1 \leqslant y^{t-1}} \frac{h(m_1)}{m_1} - \frac{w^2 y}{\log y} S_1(y^{t-1}, y) + \mathrm{O}\Big(|w|^2 \frac{r(y) y^t m(|h|, y^t)}{m(|h|, y)}\Big),$$

and

$$ty^{-t} S_1(y^t, y) - w^2 \int_0^{t-1} y^{-u} S_1(y^u, y)\,\mathrm{d}u = \mathrm{O}\Big(|w|^2 \frac{r(y) m(|h|, y^t)}{\log y\, m(|h|, y)}\Big). \tag{6}$$

Now let

$$y^{-t} S_1(y^t, y) = y^{-t} + \frac{\rho_w'(t)}{\log y} - w^2 \delta(t)\frac{y^{1-t}}{\log y} F(t, y)\frac{r(y)}{\log y}, \tag{7}$$

where

$$\delta(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t < 1, \\ 1 & \text{if} \quad t \geqslant 1. \end{cases}$$

Using (6), similarly as in [1], we find that

$$F(t,y)=\mathrm{O}\Big(|w|^2 t^{|w|^2-1}\frac{m(|h|,y^t)}{m(|h|,y)}+t^{|w|^2-1}\frac{\log^2 y}{y}\Big).$$

This and (7), since $t=\frac{\log x}{\log y}=\sqrt{\log\log x}$, yield the lemma.

Proof of the Theorem. Summing by parts, from (2) we deduce

$$\sum_{y<p\leqslant x}\frac{h(p)}{p}=w^2\log\sqrt{\log\log x}+\mathrm{O}\big(|w|^2 r(y)\log\sqrt{\log\log x}\big).$$

Therefore

$$\begin{aligned}\frac{m(h,x)}{m(h,y)}&=\prod_{y<p\leqslant x}\Big(1+\sum_{\alpha=1}^{\infty}\frac{h(p^\alpha)}{p^\alpha}\Big)=\exp\Big\{\sum_{y<p\leqslant x}\Big(\frac{h(p)}{p}+\mathrm{O}\Big(\frac{|w|^2}{p^2}\Big)\Big)\Big\}\\&=\exp\Big\{\sum_{y<p\leqslant x}\frac{h(p)}{p}\Big\}\Big(1+\mathrm{O}\Big(\frac{|w|^2}{y}\Big)\Big)\\&=\big(\sqrt{\log\log x}\big)^{w^2+\mathrm{O}(|w|^2 r(y))}\Big(1+\mathrm{O}\Big(\frac{|w|^2}{y}\Big)\Big),\end{aligned}\tag{8}$$

and

$$\frac{m(|h|,x)}{m(|h|,y)}\geqslant\Big|\frac{m(h,x)}{h,y}\Big|=\big(\sqrt{\log\log x}\big)^{\mathrm{Re}\,w^2+\mathrm{O}|w|^2 r(y)}\Big(1+\mathrm{O}\Big(\frac{|w|^2}{y}\Big)\Big).\tag{9}$$

Let positive integers m_2 be free of prime divisors greater than y, and

$$S_2(x,y)=\sum_{m_2>x}\frac{h(m_2)}{m_2},\qquad S_3(x,y)=\sum_{m_2\leqslant x}h(m_2).$$

Then in [1] it was proved that

$$S_2(x,y)=\mathrm{O}\Big(m(|h|,y)\exp\big\{-t\log t+\mathrm{O}(|w|^2 t)\big\}\Big).\tag{10}$$

$$S_3(x,y)=\mathrm{O}\Big(\frac{|w|^2 m(|h|,y)}{\log x}\exp\big\{-t\log t+\mathrm{O}(|w|^2 t)\big\}\Big).\tag{11}$$

We have

$$\sum_{m\leqslant x}h(m)=\sum_{m_2\leqslant\frac{x}{y}}h(m_2)\Big(S_1\Big(\frac{x}{m_2},y\Big)-1\Big)+\sum_{m_2\leqslant x}h(m_2).\tag{12}$$

Using Lemma 1, we find that the first sum in the right-hand side of (12) is

$$\frac{x}{\log x}\sum_{m_2\leqslant\frac{x}{y}}\frac{h(m_2)}{m_2}\rho_w'\Big(\frac{\log\frac{x}{m_2}}{\log y}\Big)-w^2\frac{y}{\log y}\sum_{m_2\leqslant\frac{x}{y}}h(m_2)$$
$$+\mathrm{O}\bigg(\Big(|w|^2x\frac{r(y)}{\log y}t^{|w|^2-1}\frac{m(|h|,x)}{m(|h|,y)}+\frac{xr(y)\log y}{y}t^{|w|^2-1}\Big)\sum_{m_2\leqslant\frac{x}{y}}\frac{|h(m_2)|}{m_2}\bigg).$$

Thus, the equality (12) can be written in the form

$$\sum_{m\leqslant x}h(m)=\frac{xm(h,y)}{\log y}\rho_w'(t)-\frac{w^2x}{\log y}\sum_{m_2>\frac{x}{y}}\frac{h(m_2)}{m_2}$$
$$-\frac{x}{\log y}\int_1^t\Big(\sum_{m_2>\frac{x}{y^u}}\frac{h(m_2)}{m_2}\Big)\rho_w''(u)\,\mathrm{d}u-\frac{w^2y}{\log y}\sum_{m_2<\frac{x}{y}}h(m_2)+\sum_{m_2\leqslant x}h(m_2)$$
$$+\mathrm{O}\Big(\frac{|w|^2xr(y)}{\log y}t^{|w|^2-1}m(|h|,x)+\frac{xr(y)\log y}{y}t^{|w|^2-1}m(|h|,x)\Big). \qquad (13)$$

By (10), (11) and (8), (9)

$$\frac{w^2x}{\log y}\sum_{m_2>\frac{x}{y}}\frac{h(m_2)}{m_2}=\mathrm{O}\Big(\frac{|w|^2xm(|h|,x)}{\log x}\exp\{-ct\}\Big),$$
$$\sum_{m_2\leqslant x}h(m_2)=\mathrm{O}\Big(\frac{|w|^2xm(|h|,x)}{\log x}\exp\{-c_1t\}\Big),$$
$$\frac{w^2y}{\log y}\sum_{m_2\leqslant\frac{x}{y}}h(m_2)=\mathrm{O}\Big(\frac{|w|^2xm(|h|,x)}{\log x}\exp\{-c_2t\}\Big).$$

The term with integral is

$$\mathrm{O}\Big(\frac{|w|^2xm(|h|,x)}{\log x}t^{-1}\Big),$$

while in view of Lemma 6 from [1] and (8)

$$\frac{xm(h,y)}{\log y}\rho_w'(t)=\frac{\mathrm{e}^{-\gamma w^2}xm(h,x)}{\Gamma(w^2)\log x}\big(\sqrt{\log\log x}\big)^{\mathrm{O}(|w|^2r(y))}\Big(1+\mathrm{O}\Big(\frac{|w|^2}{y}\Big)\Big)$$
$$+\mathrm{O}\Big(\frac{|w|^2xm(|h|,x)}{\log x}\Big)\big(\sqrt{\log\log x}^{\,-1+\mathrm{O}(|w|^2r(y))}\big).$$

All these estimates together with (13) and

$$\frac{x}{\log y}\int_1^t\Big(\sum_{m_2>\frac{x}{y^u}}\Big)\rho''_w(u)\,\mathrm{d}u = \mathrm{O}\Big(\frac{|w|^2 x m(|h|,x)}{\log x}\Big)\big(\sqrt{\log\log x}\big)^{-1}$$

prove the theorem.

References

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REZIUMĖ

R. Ivanauskaitė. Apie kai kurių parabolinių formų dzeta funkcijų koeficientų vidurkius

Gauta prabolinių formų dzeta funkcijų koeficientų vidurkio asimptotika.