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Separable extensions in Procesi category

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In this paper we define a separable centred extensions in Procesi category and give its characterizations in terms of defined derivations.

All rings in this paper are associative with identity element which should be preserved by ring homomorphisms, modules are unitary.

Let $\phi: R \to A$ be a ring homomorphism. Then A becomes a canonical R-bimodule and we write ra and ar instead of $\phi(r)a$ and $a\phi(r)$ for $r \in R$, $a \in A$. Let $Z_A(R) = \{x \in A \mid rx = xr, \forall r \in R\}$ be the set of R-centralizing elements of the ring A.

We call ϕ a *centred homomorphism* and A, a *centred extension* of R via ϕ, provided $A = RZ_A(R)$. This means that $a = \sum_k r_k x_k$ for each element $a \in A$ with some $r_k \in R$ and $x_k \in Z_A(R)$. If $Z_A(R)$ is commutative then centred extension is called a *central extension.* Rings and their centred homomorphisms form a category, known as *Procesi category.* The very important property of the Procesi category is that tensor product $A \otimes_R B$ of R extensions has a canonical ring structure and is a canonical extension of the ring R. We recall that every element in $A \otimes_R B$ can be expessed as a finite sum of the elements of the form $a \otimes y$ where $a \in A$ and $y \in Z_B(R)$. Then $(a \otimes y)(c \otimes d) = ac \otimes yd$ in $A \otimes_R B$ of R where $c \in A$ and $d \in B$.

Let R be a commutative ring, and A be an R-algebra, A° its opposite ring. Then the enveloping algebra $A^e = A \otimes_R A^\circ$ acts canonically on A from the left. Recall, see [1], that A is called separable over R if A is a projective A^e-module.

When R is noncommutative, we can not generalize the given definition of separability because if we have a ring homomorphism $\phi: R \to A$, the opposite ring A° is no longer even an R-module and there is no analogue of an enveloping algebra.

Let $\phi: R \to A$ be a centred extension, i.e., a morphism in Procesi category. Define a new ring which we also denote by A°, which as additive group coincides with A. Let $a^\circ, b^\circ \in A^\circ$ Let $a = \sum_i \alpha_i x_i$, $b = \sum_j \beta_j y_j$ with some $\alpha_i, \beta_j \in R$ and $x_i, y_j \in Z_A(R)$.

We define a new multiplication in A° as follows:

$$a^\circ b^\circ = \Big(\sum_{i,j} \alpha_i \beta_j y_j x_i\Big)^\circ = \Big(\sum_i \alpha_i b x_i\Big)^\circ = \Big(\sum_j y_j a \beta_j\Big)^\circ .$$

Equalities above show correctness of the definition of multiplication in A°. We notify that elements of the $\phi(R) \subseteq A^\circ$ are multiplied as in A. Of course, A° contains the subring $Z_A^\circ(R)$ opposite to the ring $Z_A(R)$. We call defined ring A° the *opposite centred extension.* Obviously, the ring A° canonically becomes a centred extension of the ring R. So, when R is noncommutative, multiplication in the ring A° depends from R and the homomorphism ϕ and a noncommutative situation completely differs from the case when R is commutative.

Because Procesi category is closed under tensor products, we call the ring $A^e = A \otimes_R A^\circ$ an *enveloping extension* of the centred extension ϕ: $R \to A$. Now we define a left action from A^e on A by $\lambda z = \sum_k a_k z v_k$, where $\lambda = \sum_k a_k \otimes v_k^\circ \in A^e$ with a_k, $z \in A$ and $v_k \in Z_A(R)$. Obviously this action defines the left A^e-module structure on A. It is easy to see that a canonical projection π: $A^e \to A$, sending $\sum_k a_k \otimes b_k^\circ$ to $\sum_k a_k \otimes b_k$, is the homomorphism of the left A^e modules.

We call a centred extension ϕ: $R \to A$ *separable centred extension* if A is the projective A^e-module under the given structure. So we have the exact sequence of the left A^e-modules:

$$0 \to J(A) \to A^e \xrightarrow{\pi} A \to 0.$$

Let M be a left A^e-module.So M has a canonical $A - Z_A(R)$-bimodule structure defined by $am = (a \otimes 1)m$, $my = (1 \otimes y^\circ)m$ for $a \in A$, $y \in Z_A(R)$. Evidently, if M is an $A - Z_A(R)$-bimodule having property that $\sum_k \alpha_k y_k = 0$ in A with $\alpha_k \in R$, $y_k \in Z_A(R)$, implies $\sum_k \alpha_k m y_k = 0$ in M for all $m \in M$, then M becomes a left A^e-module.

An *R-derivation* of centred extension A to a left A^e-module M we call a homomorphism ∂: $A \to M$ of the left R-modules, such that $\partial(xy) = (\partial x)y + x\partial y$, where $x, y \in Z_A(R)$ and using above noted $A - Z_A(R)$-bimodule structure on M. Using the R-linearity of ∂ we easily see that $\partial(ay) = (\partial a)y + a\partial y$, where $a \in A$, $y \in Z_A(R)$. Note, when R is commutative, this definition gives the well known definition of the derivation of the R-algebra. A derivation is called an *inner* if there exists $m \in M$ such that $\partial a = ((a \otimes 1) - (1 \otimes a^\circ))m$ for $a \in A$. Denote by $Der_R(A, M)$ an abelian group of R-derivation of A to M and by $Derin_R(A, M)$ the subgroup of the inner derivations.

THEOREM 0.1. *Homomorphism $Hom_{A^e}(J(A), M) \to Der_R(A, M)$ sending f to $f(a \otimes 1 - 1 \otimes a^\circ)$ is an isomorphism. Under this isomorphism the inner derivation correspond to $J(A)$ homomorphism extending to A^e homomorphisms to M.*

Proof. Evidently defined morphism is injective because $J(A)$ is generated over A^e by elements of the form $a \otimes 1 - 1 \otimes a^\circ$. It is surjective: let ∂: $A \to M$ be an R-derivation, define $f(a \otimes b^\circ) = -a\partial b$ for $a \otimes b^\circ \in A^e$. Evidently $f(a \otimes 1 - 1 \otimes a^\circ) = \partial a$. We leave detailed verification that defined homomorphism f is an A^e-homomorphism on $J(A)$. Second part of the theorem is obvious.

Denote $M^A = \{m \in M \mid (a \otimes 1 - 1 \otimes a^\circ)m = 0, a \in A\}$. It is clear that $M^A \cong Hom_{A^e}(A, M)$.

THEOREM 0.2. *The sequence*

$$0 \to M^A \to M \to Der_R(A, M) \to Ext_{A^e}(A, M) \to 0$$

where $M^A \to M$ is an inclusion, $M \to Der_R(A, M)$ is obtained by taking the inner derivations, is exact. In particular, $Ext_{A^e}(A, M) = H^1(A, M) \cong Der_R(A, M)/ Derin_{A^e}(A, M)$.

Proof. Exact sequence $0 \to J(A) \to A^e \xrightarrow{\pi} A \to 0$ induces the exact cohomological sequence

$$0 \to Hom_{A^e}(A, M) \to Hom_{A^e}(A^e, M) \to Hom_{A^e}(J(A), M) \to H^1(A, M) \to 0.$$

Remarks above and Theorem 0.1 proves the result.

Now we can give some characterizations of the separable extensions in Procesi category.

THEOREM 0.3. *For any centred extension the following conditions are equivalent:*

(1) *A is a projective A^e-module;*

(2) *$H^1(A, M) = 0$ for all left A^e-modules M;*

(3) *every R-derivation $\partial\colon A \to M$ is inner;*

(4) *the R-derivation $\delta\colon A \to J(A)$, $\delta a = a \otimes 1 - 1 \otimes a^\circ$ is inner.*

Proof. Equivalenence of the conditions (1) and (2) is clear. (3) follows from (2) by Theorem 0.2. Let now $\partial \in Der_R(A, M)$. By Theorem 0.1, $\partial a = f(a \otimes 1 - 1 \otimes a^\circ)$ for some $f \in Hom_{A^e}(J(A), M)$. By (3), $\delta a = (\delta a)\lambda$ with $\lambda \in J(A)$, so $\partial a = f((\delta a)\lambda) = (\delta a) f(\lambda)$, so ∂ is inner.

References

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REZIUMĖ

A. Kaučikas. Separabilūs plėtiniai Pročezio kategorijoje

Procezio kategorijoje apirbėžta separabilaus plėtinio sąvoka, kuri charakterizuota diferencijavimų terminais.