## On zeta-functions of cusp forms

Antanas LAURINČIKAS (VU, ŠU)\*

e-mail: antanas.laurincikas@maf.vu.lt

This note is a continuation of [2]. Our aim is to prove a discrete limit theorem on the complex plane  $\mathbb{C}$  for zeta-functions of certain cusp forms with step of the progression h satisfying conditions discussed in [2].

More precisely, let F(z) be a holomorphic cusp form of weight  $\kappa$  which is a normalized eigenform. Suppose that the function F(z) has the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi i m z}, \quad c(1) = 1.$$

We consider the zeta-function  $\varphi(s, F)$ ,  $s = \sigma + it$ , attached to F(z):

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad \sigma > \frac{\kappa + 1}{2}.$$

It is well known that the function  $\varphi(s, F)$  is an entire function.

Suppose that h > 0 is such that  $\exp\{\frac{2\pi k}{h}\}$  is rational for some integer  $k \neq 0$ . Denote by  $k_0$  the smallest of such k, and let  $\exp\{\frac{2\pi k_0}{h}\} = \frac{m_0}{n_0}$ ,  $m_0, n_0 \in \mathbb{N}$ ,  $(m_0, n_0) = 1$ . Let, as usual,

$$\Omega = \prod \gamma_p,$$

where  $\gamma_p$  is the unit circle  $\gamma$  on the complex plane for each prime p.  $\Omega$  is a compact topological group. Denote  $\Omega_h = \{\omega \in \Omega \colon \omega(m_0) = \omega(n_0)\}$ . Then  $\Omega_h$  is a closed subgroup of  $\Omega$ , and therefore is a compact topological group. Denote by  $\omega(p)$  the projection of  $\omega \in \Omega$  to the coordinate space  $\gamma_p$ , and let, for  $m \in \mathbb{N}$ ,

$$\omega(m) = \prod_{p^{\alpha}||m} \omega^{\alpha}(p).$$

Let  $\mathcal{B}(S)$  stand for the class of Borel sets of the space S, and denote by  $m_{hH}$  the probability Haar measure on  $(\Omega_h, \mathcal{B}(\Omega_h))$ . Then in [2] it was proved that, for  $\sigma > \kappa/2$ ,

$$\varphi(\sigma, \omega_h, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_h(m)}{m^{\sigma}}$$

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is a complex-valued random variable defined on the probability space  $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$ .

THEOREM 1. Suppose that the number h satisfies the above conditions, and  $\sigma > \frac{\kappa}{2}$ . Then the probability measure

$$P_N(A) = \frac{1}{N+1} \# (0 \leqslant m \leqslant N : \varphi(\sigma + imh, F) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the random variable  $\varphi(\sigma, \omega_h, F)$  as  $N \to \infty$ .

First the condition on the number h is used in a limit theorem for Dirichlet polynomial. Let

$$p_n(t) = \sum_{m=1}^{\infty} \frac{c(m)}{m^{\sigma + it}}, \quad n > \max(m_0, n_0).$$

Denote by  $p_1, \ldots, p_r$  distinct prime numbers which divide n!. Let

$$\Omega_r = \prod_{j=1}^r \gamma_{p_j},$$

where  $\gamma_{p_j} = \gamma$  for j = 1, ..., r, and

$$\Omega_{hr} = \{ \omega \in \Omega_r : \, \omega(m_0) = \omega(n_0) \}.$$

Consider on  $(\Omega_r, \mathcal{B}(\Omega_r))$  a probability measure

$$Q_{hN}(A) = \frac{1}{N+1} \# (0 \leqslant m \leqslant N: (p_1^{imh}, \dots, p_r^{imh}) \in A).$$

LEMMA 1. The probability measure  $Q_{hN}$  converges to the Haar measure  $m_{hr}$  on  $(\Omega_{hr}, \mathcal{B}(\Omega_{hr}))$ .

Proof. Consider the Fourier transform

$$g_N(k_1,\ldots,k_r) = \int_{\Omega} x_1^{k_1} \ldots x_r^{k_r} dQ_{hN},$$

 $(k_1,\ldots,k_r)\in\mathbb{Z}^r$ ,  $(x_1,\ldots,x_r)\in\Omega_r$ , of the measure  $Q_{hN}$ . Without loss of generality we can suppose that the prime numbers  $p_1,\ldots,p_l$  occur in the factorization of  $m_0$  and  $n_0$ , and let  $\alpha_i$  be the exponent of  $p_i$  in  $\frac{m_0}{n_0}$ ,  $i=1,\ldots,l$ . Then we have that

$$g_{N}(k_{1},...,k_{r}) = \frac{1}{N+1} \sum_{m=0}^{\infty} \prod_{j=1}^{r} p_{j}^{imhk_{j}} = \frac{1}{N+1} \sum_{m=0}^{N} \exp\{imh \sum_{j=1}^{r} k_{j} \log p_{j}\}$$

$$= \begin{cases} 1, & \text{if } k_{1} = k\alpha_{1}, ..., k_{l} = k\alpha_{l}, k_{l+1} = ... = k_{r} = 0, \\ \frac{1}{N+1} \frac{1 - \exp\{i(N+1)h \sum_{j=1}^{r} k_{j} \log p_{j}\}}{1 - \exp\{ih \sum_{j=1}^{r} k_{j} \log p_{j}\}} & \text{otherwise.} \end{cases}$$
(1)

Since the logarithms of prime numbers are linearly independent over the field of rational numbers,

$$\sum_{j=1}^{r} k_j \log p_j \neq 0 \quad \text{for } (k_1, \dots, k_r) \neq (0, \dots, 0).$$

By the definition of  $k_0$ 

$$\exp\left\{\frac{2\pi k_0}{h}\right\} = \frac{m_0}{n_0} = p_1^{\alpha_1} \dots p_l^{\alpha_l},$$

therefore the rational numbers

 $p_1^{k_1} \dots p_r^{k_r}$  and  $\exp\left\{\frac{2\pi k}{h}\right\}$  coincide if and only if  $(k_1, \dots, k_r) = (m\alpha_1, \dots, m\alpha_l, 0, \dots, 0)$ . Hence

$$\sum_{j=1}^{r} k_j \log p_j \neq \frac{2\pi k}{h} \quad \text{for any } k \in \mathbb{Z},$$

for  $(k_1, \ldots, k_l, k_{l+1}, \ldots, k_r) \neq (m\alpha_1, \ldots, m\alpha_l, 0, \ldots, 0)$ . Consequently, (1) shows that

$$\lim_{\substack{N\to\infty\\N\to\infty}} g_N(k_1,\ldots,k_r) = \begin{cases} 1, & \text{if } (k_1,\ldots,k_l,k_{l+1},\ldots,k_r) = (k\alpha_1,\ldots,k\alpha_l,0,\ldots,0), \\ 0 & \text{otherwise.} \end{cases}$$
Hence the lemma follows.

Let  $g \in \Omega_h$  and

$$p_n(t,g) = \sum_{m=1}^n \frac{c(m)g(m)}{m^{\sigma+it}}.$$

LEMMA 2. The probability measures

$$\frac{1}{N+1}\#(0\leqslant m\leqslant N\colon p_n(mh)\in A),\quad A\in\mathcal{B}(\mathbb{C}),$$

and

$$\frac{1}{N+1}\#\big(0\leqslant m\leqslant N\colon p_n(mh,g)\in A\big),\quad A\in\mathcal{B}(\mathbb{C}),$$
 both converge weakly to the same measure as  $N\to\infty$ .

*Proof.* We use Lemma 1 and the same method as in [1]. Now we define, for  $\sigma_1 > \frac{1}{2}$ ,

$$\varphi_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

and

$$\varphi_n(s,\omega_h,F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_h(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

It is not difficult to see that the later two series converge absolutely for  $\sigma > \kappa/2$ .

LEMMA 3. Suppose that  $\sigma > \frac{\kappa}{2}$ . Then there exists a probability measure  $P_n$  on  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$  such that both the measures

$$\frac{1}{N+1}\#\big(0\leqslant m\leqslant N\colon \varphi(\sigma+imh;F)\in A\big),\quad A\in\mathcal{B}(\mathbb{C}),$$

and

$$\frac{1}{N+1}\#\big(0\leqslant m\leqslant N\colon \varphi(\sigma+imh,\omega_h;F)\in A\big),\quad A\in\mathcal{B}(\mathbb{C}),$$

converge weakly to  $P_n$  as  $N \to \infty$ .

*Proof.* The proof of the lemma uses Lemma 3 and is similar to that of Lemma 3.1 of [1].

Now we give a result of the ergodic theory. Let  $a_h = \{p^{-ih}: p \text{ is prime}\}$ . We define a transformation  $f_h$  on  $\Omega_h$  by  $f_h(\omega_h) = a_h \omega_h$ ,  $\omega_h \in \Omega_h$ . Then  $f_h$  is a measurable measure preserving transformation on  $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$ .

LEMMA 4. The transformation  $f_h$  is ergodic.

*Proof.* The proof with minor changes coincides with that of Lemma 5.1 of [1]. Using Lemma 4 and the classical Birkhoff theorem, we obtain that, for  $\sigma > \frac{\kappa}{2}$ ,

$$\sum_{m=0}^{N} |\varphi(\sigma + imh, \omega_h; F|^2 = O(N), \quad N \to \infty,$$

for almost all  $\omega_h \in \Omega_h$ . The estimate

$$\sum_{m=0}^{N} |\varphi(\sigma + imh; F)|^2 = O(N), \quad N \to \infty,$$

for  $\sigma > \frac{\kappa}{2}$  follows from the estimate [3]

$$\int_0^T |\varphi(\sigma + it; F)|^2 = O(T), \quad T \to \infty.$$

The last two mean-value estimates allow us to obtain the following statement.

LEMMA 5. Let  $\sigma > \frac{\kappa}{2}$ . Then

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left| \varphi(\sigma+imh;F) - \varphi_n(\sigma+imh;F) \right| = 0$$

and

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left| \varphi(\sigma + imh, \omega_h; F) - \varphi_n(\sigma + imh, \omega_h; F) \right| = 0$$

for almost all  $\omega_h \in \Omega_h$ .

*Proof of Theorem* 1. First, using Lemmas 3 and 5, by traditional way we show that there exists a probability measure P on  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$  such that the probability measures  $P_N$  and

$$\frac{1}{N+1} \# (0 \leqslant m \leqslant N \colon \varphi(\sigma + imh, \omega_h; F) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

both converge weakly to P as  $N \to \infty$ . It remains to check that P is the distribution of the random variable  $\varphi(\sigma, \omega_h; F)$ .

Let  $A \in \mathcal{B}(\mathbb{C})$  be a continuity set of the measure P. Then by the above remark, for  $\omega_h \in \Omega_h$ ,

$$\lim_{N \to \infty} \frac{1}{N+1} \# \left( 0 \leqslant m \leqslant N \colon \varphi(\sigma + imh, \omega_h; F) \in A \right) = P(A). \tag{2}$$

Now we fix the set A and define the random variable  $\eta$  on  $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$  by the formula

$$\eta(\omega_h) = \begin{cases} 1, & \text{if } \varphi(\sigma, \omega_h; F) \in A, \\ 0, & \text{if } \varphi(\sigma, \omega_h; F) \notin A. \end{cases}$$

Then

$$E\eta = \int_{\Omega_h} \eta \, dm_{hH} = m_{hH} \big( \omega_h \in \Omega_h : \, \varphi(\sigma, \omega_h; F) \big)$$
 (3)

is the distribution of the random variable  $\varphi(\sigma, \omega_h; F)$ . However, by Lemma 4 and the Birkhoff theorem

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{m=0}^{N} \eta \left( f_h^m(\omega_h) \right) = E \eta \tag{4}$$

for almost all  $\omega_h \in \Omega_h$ . On the other hand, the definition of  $\eta$  and  $f_h$  shows that

$$\frac{1}{N+1}\sum_{m=0}^{N}\eta(f_{h}^{m}(\omega_{h}))=\frac{1}{N+1}\#(0\leqslant m\leqslant N:\varphi(\sigma+imh,\omega_{h};F)\in A).$$

This together with (3) and (4) yields

$$\lim_{N\to\infty} \frac{1}{N+1} \# (0 \leqslant m \leqslant N \colon \varphi(\sigma + imh, \omega_h; F) \in A) = P_{\varphi}(A)$$

for almost all  $\omega_h \in \Omega_h$ , where  $P_{\varphi}$  is the distribution of the random element  $\varphi(\sigma, \omega_h; F)$ . Hence and from (2) we have that

$$P(A) = P_{\varphi}(A)$$

for all continuity sets A of the measure P. From this, clearly, the theorem follows.

## References

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- 2. A. Laurinčikas, The definition of one complex-valued random variable (this volume).
- 3. K. Matsumoto, A probabilistic study on the value-distribution of Dirichlet series attached to certain cusp forms, *Nagoya Math. J.*, **116**, 123–138 (1989).

## REZIUMĖ

## A. Laurinčikas. Apie parabolinių formų dzeta funkcijas

Įrodoma diskreti ribinė teorema kompleksinėje plokštumoje parabolinių formų dzeta funkcijoms, kai progresijos žingsnis h turi savybę:  $\exp\{\frac{2\pi k}{h}\}$  yra racionalus kuriems nors  $k \neq 0$ .