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The estimate of fractional moments for Dirichlet L-functions

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Let $s=\sigma+it$ be a complex variable. Our aim is to find the estimate for fractional moments

$$I_k(\sigma,\chi)=I_k(\sigma,T,\chi)=\int_0^T \left|L(\sigma+it,\chi)\right|^{2k}\mathrm{d}t$$

of Dirichlet L-functions. Here χ is a primitive character mod q, and $k=\frac{1}{n}$, $n=1,2,3,\ldots$. We prove the following theorem.

THEOREM 1. *Let $T\to\infty$. Then the estimate*

$$I_k\left(\frac{1}{2},T,\chi\right)\ll T(\log T)^{k^2}$$

holds.

From the Euler product it follows that, for $\sigma>1$,

$$L^k(s,\chi)=\exp\bigl(k\log(L(s,\chi))\bigr)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-k}.$$

Hence we find that

$$L^k(s,\chi)=\prod_p\sum_{j=0}^{\infty}\frac{d_k(p^j)\chi^j(p)}{p^{js}}=\sum_{n=1}^{\infty}\frac{d_k(n)\chi(n)}{n^s},$$

where

$$d_k(p^r)=\frac{\Gamma(k+r)}{\Gamma(k)r!}=\frac{k(k+1)\ldots(k+r-1)}{r!}.$$

LEMMA 1. *For a fixed real number $k\geqslant 0$ there exists a constant $c_k>0$ such that uniformly in σ, $\frac{1}{2}+\frac{c_k}{\log(N)}\leqslant\sigma\leqslant 1$, the estimates*

$$\sum_{n=1}^{N}\frac{d_k(n)^2|\chi(n)|^2}{n^{2\sigma}}\ll\left(\sigma-\frac{1}{2}\right)^{-k^2}\quad\textit{and}\quad\sum_{n=1}^{N}\frac{d_k(n)^2|\chi(n)|^2}{n}\ll(\log(N))^{k^2}$$

are valid.

Proof. Taking $\sigma = \frac{1}{2} + \frac{c_k}{\log(N)}$, we obtain that $n^{-2\sigma} \ll n^{-1}$ when $1 \leqslant n \leqslant N$. Therefore the second inequality of the lemma follows from the first inequality. At first we notice that the inequality

$$\left(\frac{\Gamma(k+r)}{\Gamma(k)r!}\right)^2 \leqslant \frac{\Gamma(k^2+r)}{\Gamma(k^2)r!}$$

can be proved by mathematical induction. Therefore $d_k(n)^2 \leqslant d_{k^2}(n)$, and hence it follows that

$$\sum_{n=1}^{N} \frac{d_k(n)^2|\chi(n)|^2}{n^{2\sigma}} \leqslant \sum_{n=1}^{N} \frac{d_k(n)^2}{n^{2\sigma}} \leqslant \sum_{n=1}^{N} \frac{d_{k^2}(n)}{n^{2\sigma}} = \zeta^{k^2}(2\sigma) \ll \left(\sigma - \frac{1}{2}\right)^{-k^2}.$$

LEMMA 2. *Let $f(z)$ be a regular function in the strip $\alpha < \mathrm{Re}(z) < \beta$ and continuous for $\alpha \leqslant \mathrm{Re}(z) \leqslant \beta$. Suppose that $f(z) \to 0$ when $|\Im(z)| \to \infty$ uniformly for $\alpha \leqslant \mathrm{Re}(z) \leqslant \beta$. Then for $\alpha \leqslant \gamma \leqslant \beta$ and all $m > 0$*

$$\int_{-\infty}^{\infty} |f(\gamma+it)|^m \,\mathrm{d}t \leqslant \left(\int_{-\infty}^{\infty} |f(\alpha+it)|^m \,\mathrm{d}t\right)^{\frac{\beta-\gamma}{\beta-\alpha}} \left(\int_{-\infty}^{\infty} |f(\beta+it)|^m \,\mathrm{d}t\right)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

Proof of the lemma can be found in [2].

Let

$$w(t) = \int_T^{2T} \exp\big(-2k(t-\tau)^2\big)\,\mathrm{d}\tau, \quad J(\sigma,\chi) = \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k} w(t)\,\mathrm{d}t. \quad (1)$$

LEMMA 3. *Let $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$ and $T \geqslant 2$. Then*

$$J\left(\frac{1}{2},\chi\right) \ll T^{k(\sigma-\frac{1}{2})} J(\sigma,\chi) + \mathrm{e}^{\frac{-kT^2}{3}}.$$

Proof. We take in the Lemma 2 $f(z) = f(z,\chi) = L(z,\chi)\exp((z-i\tau)^2)$, $\gamma = \frac{1}{2}$, $\alpha = 1-\sigma$, $\beta = \sigma$ where $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$, $m = 2k > 0$. Then by the functional equation for L-functions [3] we obtain that

$$L(\alpha+it,\chi) \ll |L(\beta+it,\chi)|(1+|t|)^{\sigma-\frac{1}{2}}.$$

From this it follows

$$\int_{-\infty}^{\infty} |f(\alpha+it)|^{2k} \,\mathrm{d}t \ll \tau^{2k}\mathrm{e}^{\frac{-k\tau^2}{2}} + \tau^{k(2\sigma-1)} \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k} \mathrm{e}^{-2k(t-\tau)^2}\,\mathrm{d}t.$$

Then from Lemma 2 we have

$$\int_{-\infty}^{\infty} \left|L\left(\frac{1}{2}+it,\chi\right)\right|^{2k} \mathrm{e}^{-2k(t-\tau)^2}\,\mathrm{d}t$$
$$\ll \mathrm{e}^{\frac{-2k\tau^2}{5}} + \tau^{k(\sigma-\frac{1}{2})} \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k} \mathrm{e}^{-2k(t-\tau)^2}\,\mathrm{d}t.$$

Now the integration over $[T, 2T]$ completes the proof of the lemma.

LEMMA 4. *Let* $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$ *and* $T \geqslant 2$. *Then*

$$J(\sigma, \chi) \ll T^{\sigma - \frac{1}{2}} J\left(\frac{1}{2}, \chi\right) + \mathrm{e}^{\frac{-kT^2}{4}}.$$

Proof. We take in Lemma 2 $f(z) = f(z, \chi) = L(z, \chi)\exp((z - i\tau)^2)$, $\gamma = \sigma$, $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $m = 2k > 0$, where $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$. Then we obtain

$$\int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t \ll \tau^{2k} \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^{2k} \mathrm{e}^{-2k(t-\tau)^2}\, \mathrm{d}t + \mathrm{e}^{-\frac{2k\tau^2}{5}}.$$

Similarly we find that

$$\int_{-\infty}^{\infty} \left| f\left(\frac{3}{2} + it\right)\right|^{2k} \mathrm{d}t \ll \tau^{2k} \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} \mathrm{e}^{-2k(t-\tau)^2}\, \mathrm{d}t + \mathrm{e}^{-\frac{2k\tau^2}{5}} \ll \tau^{2k}.$$

Lemma 2 implies

$$\int_{-\infty}^{\infty} |f(\sigma + it)|^{2k}\, \mathrm{d}t \ll \tau^{2k} \left(\int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^{2k} \mathrm{e}^{-2k(t-\tau)^2}\, \mathrm{d}t \right)^{\frac{3}{2} - \sigma} + \mathrm{e}^{-\frac{k\tau^2}{3}}, \quad (2)$$

therefore the estimate

$$\int_{-\infty}^{\infty} |L(\sigma + it, \chi)|^{2k} \mathrm{e}^{-2k(t-\tau)^2}\, \mathrm{d}t \ll \tau^{-2k} \int_{-\infty}^{\infty} |f(\sigma + it)|^{2k}\, \mathrm{d}t + \mathrm{e}^{-\frac{2k\tau^2}{5}}$$

follows. Finally, the integrating over $[T, 2T]$ and combining with estimate (2) give the estimate of the lemma.

Now let $N = T^{\frac{1}{2}}$, and define

$$S(s, \chi) = \sum_{n=1}^{N} \frac{d_k(n)\chi(n)}{n^s},$$

$$g(s, \chi) = L(s, \chi) - S^{\frac{1}{k}}(s, \chi), \quad (3)$$

$$K(\sigma, \chi) = \int_{-\infty}^{\infty} |g(\sigma + it, \chi)|^{2k} w(t)\, \mathrm{d}t.$$

LEMMA 5. *Suppose that* $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$, $\varepsilon > 0$ *and* $T \geqslant 2$. *Then*

$$K(\sigma, \chi) \ll K\left(\frac{1}{2}, \chi\right)^{\frac{5-4\sigma}{3}} \left(TN^{\frac{\varepsilon - \frac{3}{2}}{n}}\right)^{\frac{4\sigma-2}{3}} + K\left(\frac{1}{2}, \chi\right)^{\frac{7-8\sigma}{3}} \mathrm{e}^{-\frac{4kT^2(2\sigma-1)}{6}}.$$

Proof. In Lemma 2 we take $f(z) = f(z, \chi) = g(z, \chi)\exp((z - i\tau)^2)$, $\gamma = \sigma$, $\alpha = \frac{1}{2}$, $\beta = \frac{7}{8}$ and $m = 2k = \frac{2}{n}$. Then we obtain

$$\int_{-\infty}^{\infty} |f(\sigma + it)|^{\frac{2}{n}} \mathrm{d}t \leqslant \left\{ \int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t \right\}^{\frac{7-8\sigma}{3}} \left\{ \int_{-\infty}^{\infty} \left| f\left(\frac{7}{8} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t \right\}^{\frac{8\sigma-4}{3}}. \quad (4)$$

Now we observe that, for $\frac{1}{2} \leqslant \mathrm{Re}\,(s) \leqslant 2$, we have $S(s, \chi) \ll N \ll T$ and

$$g(s, \chi) \ll (T + |t|)^n,$$

where $|s - 1| \geqslant \frac{1}{10}$. Then

$$\int_{-\infty}^{\infty} \left| f\left(\frac{7}{8} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t \ll \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} \left| f\left(\frac{7}{8} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t + (T + |t|)^{2+2k} \mathrm{e}^{-\frac{2k\tau^2}{3}}. \quad (5)$$

The integral in the right-hand side of (5) can be estimated by using Gabriel's theorem [1, Theorem 1]. From (4) and (5) we obtain

$$\int_{-\infty}^{\infty} |f(\sigma + it)|^{\frac{2}{n}} \mathrm{d}t \leqslant \left(\int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t \right)^{\frac{5-4\sigma}{3}} \left(\int_{-\infty}^{\infty} \left| f\left(\frac{5}{4} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t \right)^{\frac{4\sigma-2}{3}}$$
$$+ \left(\int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right) \right|^{\frac{2}{n}} \mathrm{d}t \right)^{\frac{7-8\sigma}{3}} \left(T^{2+2k} \mathrm{e}^{-\frac{k\tau^2}{7}} \right)^{\frac{8\sigma-4}{3}}.$$

Now we integrate over $T \leqslant \tau \leqslant 2T$, and by using (1) and (3) we deduce

$$K(\sigma, \chi) \ll K\left(\frac{1}{2}, \chi\right)^{\frac{5-4\sigma}{3}} \left(\int_{T}^{2T} \left| g\left(\frac{5}{4} + it, \chi\right) \right|^{2k} \mathrm{d}t \right)^{\frac{4\sigma-2}{3}} + K\left(\frac{1}{2}, \chi\right)^{\frac{7-8\sigma}{3}} \mathrm{e}^{-\frac{4T^2(2\sigma-1)}{6}}. \quad (6)$$

Next we have that $g(s, \chi) = \sum_{n=N}^{\infty} \frac{a_n \chi(n)}{n^s}$, where $\sigma > 1$ and $0 \leqslant a_n \leqslant d_k(n)$. Thus, by the Montgomery–Vaughan mean value theorem [3] we find

$$\int_{\frac{T}{2}}^{3T} \left| g\left(\frac{5}{4} + it, \chi\right) \right|^2 \mathrm{d}t \ll T \sum_{n=N}^{\infty} \frac{|a_n|^2 |\chi(n)|^2}{n^{\frac{5}{2}}} + \sum_{n=N}^{\infty} \frac{|a_n|^2 |\chi(n)|^2}{n^{\frac{3}{2}}} \ll T N^{\varepsilon - \frac{3}{2}}.$$

From this and (6) the lemma follows.

Proof of the Theorem 1. Let, for $\frac{1}{2} \leqslant \sigma < \frac{3}{4}$, $Y(\sigma, \chi) = \int_{-\infty}^{\infty} |S(\sigma + it, \chi)|^2 w(t)\,\mathrm{d}t$. We have $w(t) \ll \exp\left(-\frac{(T^2+t^2)k}{18}\right)$ for $t \leqslant 0$, $t \geqslant 3T$ and $S(\sigma + it, \chi) \ll T$. Furthermore, for $\frac{4T}{3} \leqslant t \leqslant \frac{5T}{3}$, $w(t)$ is bounded. By the Montgomery–Vaughan theorem [3] we find

$$\int_{0}^{3T} |S(\sigma + it, \chi)|^2 \mathrm{d}t \ll T \sum_{n=1}^{N} \frac{d_k(n)^2 |\chi(n)|^2}{n^{2\sigma}} \ll T \sum_{n=1}^{N} \frac{d_k(n)^2}{n^{2\sigma}}.$$

Therefore from Lemma 1 it follows

$$Y(\sigma,\chi) \ll T\Big(\sigma-\frac{1}{2}\Big)^{-k^2},$$

and, for $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$,

$$Y\Big(\frac{1}{2},\chi\Big) \ll T(\log T)^{k^2}. \tag{7}$$

We trivially have $|S(\sigma+it,\chi)|^2 \ll |L(\sigma+it,\chi)|^{\frac{2}{n}} + |g(\sigma+it,\chi)|^{\frac{2}{n}}$, therefore $Y(\sigma,\chi) \ll J(\sigma,\chi) + K(\sigma,\chi)$. Similarly we deduce

$$J(\sigma,\chi) \ll Y(\sigma,\chi) + K(\sigma,\chi) \tag{8}$$

and

$$K\Big(\frac{1}{2},\chi\Big) \ll Y\Big(\frac{1}{2},\chi\Big) + J\Big(\frac{1}{2},\chi\Big).$$

From estimates (7) and (8) we find that

$$J\Big(\frac{1}{2},\chi\Big) \ll T(\log T)^{k^2}. \tag{9}$$

For $K\big(\frac{1}{2},\chi\big) \geqslant T$, we can obtain that $J\big(\frac{1}{2},\chi\big) \ll Y(\sigma,\chi) + Y\big(\frac{1}{2},\chi\big) \ll T(\log T)^{k^2}$. Since $w(t) \ll 1$ for all t and the estimate $w(t) \ll \exp\big(-\frac{k(t^2+T^2)}{18}\big)$ is valid when $t \leqslant 0$ and $t \geqslant 3T$, we deduce that

$$J\Big(\frac{1}{2},\chi\Big) \ll I_k\Big(\frac{1}{2},3T,\chi\Big) + \Big(\int_{-\infty}^{0} + \int_{3T}^{\infty}\Big) e^{-\frac{k(t^2+T^2)}{19}}\,dt \ll I_k\Big(\frac{1}{2},3T,\chi\Big) + e^{-\frac{kT^2}{19}}.$$

Moreover, $w(t) \gg 1$ if $\frac{4T}{3} \leqslant t \leqslant \frac{5T}{3}$, therefore from (9) we find that

$$I_k\Big(\frac{1}{2},\frac{5T}{3},\chi\Big) - I_k\Big(\frac{1}{2},\frac{4T}{3},\chi\Big) \ll J\Big(\frac{1}{2}+it,\chi\Big) \ll T(\log T)^{k^2}.$$

Replacing T by $\big(\frac{4}{5}\big)^n T$ and summing over n, we deduce the estimate of the theorem.

References

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REZIUMĖ

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