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Constructing regular n-angles from arbitrary scalene n-angles

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1. Introduction

This note is a story about "an'tiquities" but I hope that it contains an interesting result (so far not published). It all began (and unfortunately, finished soon) in 1963 when prof. Kleopas Grincevičius proposed to first-year students of Vilnius University the following task from plane geometry:

It is well known that if on each side of any (scalene) triangle equilateral triangles are constructed (outwards, inwards, respectively), then the join of their centers forms an equilaterial triangle. Moreover, if in the same way on each side of any quadrangle squares are constructed, then the join of their centers forms a quadrangle with equal and perpendicular diagonals. The question arising in this context was the following:

QUESTION 1.1 (prof. K. Grincevičius). *If on each side of any pentagon equilaterial regular pentagons are constructed, then what properties has the join of their centers?*

In the same year I obtained an answer to this question for arbitrary n-angles. To formulate this more general result properly, some discussion is needed. Let $A_1, \ldots, A_n$ be an arbitrary scalene n-angle (with $n \geqslant 3$), i.e., a finite ordered sequence of points in a plain. First, it is vague meaning of the construction on each side of the n-angle $A_1 \ldots, A_n$ regular n-angles outwards or inwards (if the base n-angle $A_1, \ldots, A_n$ has intersections of sides). Fortunately, this is not mater, as one can build regular n-angles on the left side (on the right side, respectively) with respect to the circuit direction $A_1, A_2, \ldots, A_n, A_1, A_2, \ldots$ Second, we have not only convex regular n-angle (with the central angle $360°/n$), but also the star-like regular n-angles with the following central angles: $(2/n)360°$, $(3/n)360°, \ldots, ([n-1/2]/n)360° \neq m180°$ (degenerate case), where m is some integer, and $[n-1/2]$ denotes the integer part of $n-1/2$. Fortunately, one can build regular n-angles of the same type, i.e., with the same fixed central angle.

2. An answer to the question of prof. K. Grincevičius

THEOREM 2.1. *Let $A_1 \ldots A_n (n \geqslant 3)$ be an arbitrary scalene n-angle in the Euclidean plane. Let $O_1 \ldots O_n$ be the centers of equilateral regular n-angles of the same type constructed on each side of $A_1 \ldots A_n$ and equally oriented with respect to circuit* *direction $A_1 \dots A_n A_1 \dots$ (i.e., O_1 the center of the figure builded on the side A_1A_2, O_2 – on A_2A_3, ...). Define $n-1$ points $X_1, \dots, X_{n-1}$ as the following linear combinations:*

$$X_k = \sum_{i=1}^{[\frac{n-1}{2}]} \sin i\alpha (O_{\underline{k+i}} - O_{\underline{k+n-i}}), \quad k = 1, \dots, n-1, \tag{1}$$

where α is a certain (selective) central angle of the equilateral regular n-angles, i.e., one of

$$\frac{1}{n}360°, \ \frac{2}{n}360°, \dots, \frac{[\frac{n-1}{2}]}{n}360° \neq 180° m,$$

$$\underline{k+i} = \begin{cases} k+i, & \text{if } k+i \leqslant n, \\ k+i-n, & \text{if } k+i > n, \end{cases}$$

and $[m]$ denotes the integer part of a number m. Then the following points

$$X_1, X_1 + X_2, \dots, X_1 + \cdots X_{n-1} \quad \text{and } 0$$

are vertices of equilateral regular n-angle with the central angle α.

Proof. Define

$$X_k = \sum_{i=1}^{n-1} a_i (O_{\underline{k+i}} - O_k), \quad k = 1, \dots, n, \tag{2}$$

for some scalars $a_1, \dots, a_{n-1}$. It is clear that $X_1 + \cdots + X_n = 0$, i.e., these vectors form a closed figure. We will intend to find scalars such that this figure would become regular. Let $\mathbf{m}$ be the basis vector perpendicular to the plane (of $A_1 \dots A_n$). Let B be the midpoint of the side A_1A_2. Then the vectors $[(B - A_1) \times \mathbf{m}]$ and $O_1 - B$ are collinear. We have that

$$O_1 - B = \operatorname{ctg}\frac{\alpha}{2}[(B - A_1) \times \mathbf{m}] = \operatorname{ctg}\frac{\alpha}{2}\left[\frac{A_2 - A_1}{2} \times \mathbf{m}\right].$$

Let the direction of $\mathbf{m}$ be such that $\operatorname{ctg}\frac{\alpha}{2} > 0$ (note that $\operatorname{ctg}\frac{\alpha}{2} \neq 0$ since B never coincide with O_1). Hence, we have that

$$O_1 = \frac{A_2 + A_1}{2} + \operatorname{ctg}\frac{\alpha}{2}\left[\frac{A_2 - A_1}{2} \times \mathbf{m}\right].$$

Anologously, it can be shown that

$$O_2 = \frac{A_3 + A_2}{2} + \operatorname{ctg}\frac{\alpha}{2}\left[\frac{A_3 - A_2}{2} \times \mathbf{m}\right],$$

....

$$O_n = \frac{A_1 + A_n}{2} + \operatorname{ctg}\frac{\alpha}{2}\left[\frac{A_1 - A_n}{2} \times \mathbf{m}\right].$$

Let $X_1 \dots X_n$ be points such that the points X_1, $X_1 + X_2$, ... , $X_1 + X_2 + \cdots + X_{n-1}$ and 0 are vertices of a regular n-angle. We have that

$$X_2 = e_1[X_1 \times \mathbf{m}] + e_2 X_1$$

for some scalars e_1 and e_2. Since we have supposed that vectors $X_1, X_2, \dots, X_n$ form a regular figure, it follows that

$$\begin{aligned} X_1 &= -\sin\alpha[X_n \times \mathbf{m}] + \cos\alpha X_n, \\ X_2 &= -\sin\alpha[X_1 \times \mathbf{m}] + \cos\alpha X_1, \\ &\dots\dots\dots \\ X_n &= -\sin\alpha[X_{n-1} \times \mathbf{m}] + \cos\alpha X_{n-1}, \end{aligned}$$

where α is a central angle of the regular n-angle. Then, in view of (2),

$$\begin{aligned} &\sum_{i=1}^{n-1} a_i \left(A_{\underline{i+3}} + A_{\underline{i+2}} - A_2 - A_3 + \operatorname{ctg}\frac{\alpha}{2}\left[(A_{\underline{i+3}} - A_{\underline{i+2}} - A_3 + A_2) \times \mathbf{m}\right]\right) \\ &= -\sin\alpha \sum_{i=1}^{n-1} a_i \left(\operatorname{ctg}\frac{\alpha}{2}(-A_{\underline{i+2}} + A_{i+1} + A_2 - A_1) + \left[(A_{\underline{i+2}} + A_{i+1} - A_1 - A_2) \times \mathbf{m}\right]\right) \\ &+ \cos\alpha \sum_{i=1}^{n-1} a_i \left(A_{\underline{i+2}} + A_{i+1} - A_2 - A_1 + \operatorname{ctg}\frac{\alpha}{2}\left[(A_{\underline{i+2}} - A_{i+1} - A_2 + A_1) \times \mathbf{m}\right]\right). \end{aligned}$$

It follows that

$$\begin{aligned} &A_1(a_{n-2} + a_{n-1}) - A_2 \sum_{i=1}^{n-2} a_i - A_3 \sum_{i=2}^{n-1} a_i + \sum_{i=4}^{n} A_i(a_{i-3} + a_{i-2}) \\ &+ \operatorname{ctg}\frac{\alpha}{2}\Bigg\{[A_1 \times \mathbf{m}](a_{n-2} - a_{n-1}) + [A_2 \times \mathbf{m}]\Big(a_{n-1} + \sum_{i=1}^{n-1} a_i\Big) \\ &- [A_3 \times \mathbf{m}]\Big(a_1 + \sum_{i=1}^{n-1} a_i\Big) + \sum_{i=4}^{n}[A_i \times \mathbf{m}](a_{i-3} - a_{i-2})\Bigg\} \\ &= A_1\Bigg(\sin\alpha \operatorname{ctg}\frac{\alpha}{2}\Big(\sum_{i=1}^{n-1} a_i + a_{n-1}\Big) - \cos\alpha \sum_{i=1}^{n-2} a_i\Bigg) \\ &+ A_2\Bigg(-\sin\alpha \operatorname{ctg}\frac{\alpha}{2}\Big(a_1 + \sum_{i=1}^{n-1} a_i\Big) - \cos\alpha \sum_{i=2}^{n-1} a_i\Bigg) \\ &- \sin\alpha \operatorname{ctg}\frac{\alpha}{2} \sum_{i=3}^{n} A_i(-a_{i-2} + a_{i-1}) + \cos\alpha \sum_{i=3}^{n} A_i(a_{i-2} + a_{i-1}) \end{aligned}$$

$$+[A_1\times\mathbf{m}]\Big(\sin\alpha\sum_{i=1}^{n-2}a_i+\cos\alpha\operatorname{ctg}\frac{\alpha}{2}\Big(\sum_{i=1}^{n-1}a_i+a_{n-1}\Big)\Big)$$

$$+[A_2\times\mathbf{m}]\Big(\sin\alpha\sum_{i=2}^{n-1}a_i-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}\Big(a_1+\sum_{i=1}^{n-1}a_i\Big)\Big)$$

$$-\sum_{i=3}^{n}[A_i\times\mathbf{m}]\Big(\sin\alpha(a_{i-2}+a_{i-1})-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(a_{i-2}-a_{i-1})\Big).$$

Now we equate coefficients at respective vectors and obtain the following $2n$ equations:

$$\sin\alpha\operatorname{ctg}\frac{\alpha}{2}(a_1+a_2+\cdots+a_{n-2}+2a_{n-1})-\cos\alpha(a_1+\cdots+a_{n-2})=a_{n-2}+a_{n-1},$$

$$\sin\alpha\operatorname{ctg}\frac{\alpha}{2}(2a_1+a_2+\cdots+a_{n-1})+\cos\alpha(a_2+\cdots+a_{n-1})=a_1+\cdots+a_{n-2},$$

$$\sin\alpha\operatorname{ctg}\frac{\alpha}{2}(-a_1+a_2)-\cos\alpha(a_1+a_2)=a_2+a_3+\cdots+a_{n-1},$$

$$\sin\alpha\operatorname{ctg}\frac{\alpha}{2}(a_2-a_3)+\cos\alpha(a_2+a_3)=a_1+a_2,$$

$$\sin\alpha\operatorname{ctg}\frac{\alpha}{2}(a_3-a_4)+\cos\alpha(a_3+a_4)=a_2+a_3,$$

....

$$\sin\alpha\operatorname{ctg}\frac{\alpha}{2}(a_{n-2}-a_{n-1})+\cos\alpha(a_{n-2}+a_{n-1})=a_{n-3}+a_{n-2},$$

$$\sin\alpha(a_1+a_2+\cdots+a_{n-2})+\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(a_1+\cdots+a_{n-2}+2a_{n-1})$$
$$=\operatorname{ctg}\frac{\alpha}{2}(a_{n-2}-a_{n-1}),$$

$$\sin\alpha(a_2+\cdots+a_{n-1})-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(2a_1+a_2+\cdots+a_{n-1})$$
$$=\operatorname{ctg}\frac{\alpha}{2}(a_1+\cdots+a_{n-2}+2a_{n-1}),$$

$$\sin\alpha(a_1+a_2)-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(a_1-a_2)=\operatorname{ctg}\frac{\alpha}{2}(2a_1+a_2+\cdots+a_{n-1}),$$

$$\sin\alpha(a_2+a_3)-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(a_2-a_3)=\operatorname{ctg}\frac{\alpha}{2}(a_2-a_1),$$

$$\sin\alpha(a_3+a_4)-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(a_3-a_4)=\operatorname{ctg}\frac{\alpha}{2}(a_3-a_2),$$

....

$$\sin\alpha(a_{n-2}+a_{n-1})-\cos\alpha\operatorname{ctg}\frac{\alpha}{2}(a_{n-2}-a_{n-1})=\operatorname{ctg}\frac{\alpha}{2}(a_{n-2}-a_{n-3}).$$

Next, the sum $a_1 + a_2 + \cdots + a_{n-1}$ setting equal to zero, in view of the identities

$$\sin\alpha \operatorname{ctg}\frac{\alpha}{2} = 1 + \cos\alpha \quad \text{and} \quad \sin\alpha \tan\frac{\alpha}{2} = 1 - \cos\alpha,$$

the enlarged system of equations reduces to

$$\begin{aligned}
&a_1 + a_2 + \ldots + a_{n-1} = 0,\\
&a_{n-2} = 2\cos\alpha a_{n-1},\\
&a_1 = -a_{n-1},\\
&a_2 = 2\cos\alpha a_1,\\
&a_3 = 2\cos\alpha a_2 - a_1,\\
&a_4 = 2\cos\alpha a_3 - a_2,\\
&\ldots\ldots\ldots\\
&a_{n-1} = 2\cos\alpha a_{n-2} - a_{n-3},\\
&a_{n-2} = 2\cos\alpha a_{n-1},\\
&a_{n-1} = -a_1,\\
&a_2 = 2\cos\alpha a_1,\\
&a_3 = 2\cos\alpha a_2 - a_1,\\
&a_4 = 2\cos\alpha a_3 - a_2,\\
&\ldots\ldots\ldots\\
&a_{n-1} = 2\cos\alpha a_{n-2} - a_{n-3}.
\end{aligned}$$

The routine check shows that this system of equations has the following solution:

$$a_k = \sin k\alpha, \quad k = 1, 2, \ldots, n-1.$$

Note that

$$\begin{aligned}
&\sin\alpha = -\sin(n-1)\alpha,\\
&\sin 2\alpha = -\sin(n-2)\alpha,\\
&\ldots\ldots\ldots
\end{aligned}$$

Moreover, in the case when n is even, $\sin\frac{n}{2}\alpha = 0$. This completes the proof.

REZIUMĖ

R.P. Gylys. Konstruojant taisyklingus n-kampius iš bet kokių nelygiakraščių n-kampių

Aprašomas įdomus uždavinėlis iš planimetrijos, kurį 1963 metais išsprendžiau vadovaujant profesoriui Kleopui Grincevičiui.