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Resolution method for some class of formulas of modal logic S4

Stanislovas NORGĖLA (VU)
e-mail: stasys.norgela@maf.vu.lt

We will consider the sequents $F_1, ..., F_n \vdash$, in which F_i $i = 1, ..., n$ are the formulas of quantified modal logic having the following form:

$$\forall^* \Box^i \forall^* \exists^* (A_1 \vee ... \vee A_s \vee \Box B_1 \vee ... \vee \Box B_u \vee \Diamond C_1 \vee ... \vee \Diamond C_v), \tag{1}$$

where $A_1, B_1, C_1, A_2, B_2, C_2, ...$ are literals of classical logic. $i = 0, 1$. $\Box^1 F = \Box F$. $\Box^0 F = F$. In addition, the formulas (1) can contain the constants. G.Mints described in [1] a reduction of an arbitrary formula F of quantified modal logic to a finite set of such formulas $G_1, ..., G_m$, that $\vdash F$ is derivable in S4 if and only if $G_1, ..., G_m \vdash$ is derivable in S4. The formulas under consideration (1) have more general form.

We skolemize the formulas (1) using the method described in [2]. We delete every existential quantifier $\exists x$ and replace every occurrence of x by its functional term, which has the form $f_x^n(y_1, ..., y_m)$, where n is modal degree of the variable x in the formula and $y_1, ..., y_m$ are all the universal variables such that $\exists x$ is in the scope of $\forall y_i$. Using the obtained terms we introduce a set of ground terms. We denote by H_j^i $i = 0, 1$; $j \geqslant 0$ (see [2], [3]) a set of ground terms in which the modal degree of terms does not exceed i and the height of terms does not exceed j. In addition we assume that a set H_0^i contains a new constant a of modal degree 0 not occuring in an initial formula.

The resolution system consists of the substitution, simplification, duplication, factorization rules and of the rules for logical connectives and modal operators.

Substitution rules:

$$(S1)\quad \frac{\forall x_1 ... \forall x_m \Box \forall x_{m+1} ... \forall x_n F(x_1, ..., x_m, x_{m+1}, ..., x_n)}{\Box F(t_1, ..., t_m, t_{m+1}, ..., t_n)},$$

$t_1, ..., t_m \in H_j^0$; $t_{m+1}, ..., t_n \in H_j^1$,

$$(S2)\quad \frac{\forall x_1 ... \forall x_n F(x_1, ..., x_n)}{F(t_1, ..., t_n)},$$

$t_1, ..., t_n \in H_j^0$.

A formula F in the rules (S1), (S2) is a quantifier-free formula.

The rules for logical connectives and modal operators, the simplification, duplication and factorization rules are the same as in [4]. They are applicable only to the formulas not containing conjunction. For example, the rules for modal operators in the case under consideration have the following form:

$$(m1)\ \frac{res(\Box F, \Box G)}{\Box res(F, G)}, \qquad (m2)\ \frac{res(\Box F, \Diamond G)}{\Diamond res(F, G)}, \qquad (m3)\ \frac{res(\Box F, G)}{res(F, G)}.$$

THEOREM 1. *A set S of formulas of the form (1) is refutable if and only if the empty clause is derivable from S.*

Proof. According to Theorem 1 of [3] a set of formulas of the form (1) $S = \{G_1, ..., G_n\}$ is refutable if and only if exists such j, that the set of propositional Herbrand j-expansions is S4-contradictory. This means that a sequent $G_1, ..., G_n \vdash$ is derivable in S4. Thus, we reduce an establishment of inconsistency of the set S to an establishment of inconsistency of a set S'_j of propositional formulas. We can obtain all propositional formulas from j-th expansions using only two substitution rules (see [3]). The obtained set S'_j of propositional formulas is S4-contradictory if and only if the empty clause is derivable from S'_j. Theorem is proved.

We shall study of form of formulas obtained from S'_j by using the resolution rules. The set S'_j contains the propositional formulas having the following form

$$\Box^i (A_1 \vee ... \vee A_s \vee \Box B_1 \vee ... \vee \Box B_u \vee \Diamond C_1 \vee ... \vee \Diamond C_v), \tag{2}$$

where $i = 0, 1$; A_l, B_l, C_l are literals of propositional classical logic. We use $\alpha, \alpha_i, \alpha'_i, ...$ for arbitrary formulas of the form

$$A_1 \vee ... \vee A_s \vee \Box B_1 \vee ... \vee \Box B_u \vee \Diamond C_1 \vee ... \vee \Diamond C_v,$$

that is, for the formulas of the form (2) with $i = 0$.

THEOREM 2. *It suffies to consider for establishment of inconsistency of a set S'_j only the formulas derivable from S'_j and having the shape of*

$$\alpha \vee \Box\alpha_1 \vee ... \vee \Box\alpha_m \vee \Diamond\Box D_1 \vee ... \vee \Diamond\Box D_k, \tag{3}$$

where D_i are literals of classical logic.

Proof. Note that the formulas (3) do not contain *res*. The initial formulas are obtained by using only the substitution rules and they are of required form.

We will now show that all formulas not containing *res* derivable from S'_j have the same as (3) form. It is easy to see that after having applied the factorization or simplification rules we obtain the formulas of the form (3) also. We shall now consider in more detail the formulas obtained by applying the modal and classical (for logical connectives) rules.

Assume we have the formulas F, G of the form (3) for which the factorization or simplification rules are not applicable. In the case under consideration only the rule of introduction of *res* is applicable.

$$\frac{F,\quad G}{res(F,G)}$$

We obtain a generalized formula (see [4]) $res(F,G)$. We can derive a formula from $res(F,G)$, not containing *res*, in the case where the formulas F, G contain a complementary pair. In that follows, we assume that $\neg\neg H = H$.

Case 1. A complementary pair has the form $l, \neg l$, where l is a classical literal. Moreover, the occurrences under consideration $l, \neg l$ do not occur in a modal literal.
Let

$$F = \alpha' \vee \Box\alpha'_1 \vee \ldots \vee \Box\alpha'_m \vee \beta',$$
$$G = \alpha'' \vee \Box\alpha''_1 \vee \ldots \vee \Box\alpha''_n \vee \beta''.$$

We use β', β'' for arbitrary formulas of the form $\Diamond\Box D_i$. Three subcases are possible.

a) One of complementary literal (suppose that this is l) occurs in α' (that is, $\alpha' = l \vee \alpha'_0$) and another in α'' (that is, $\alpha'' = \neg l \vee \alpha''_0$),
b) $l \in \alpha''$ and $\neg l$ occurs in any formula $\Box\alpha''_i$ (let $\Box\alpha''_i$ be $\Box\alpha''_1$),
c) $l \in \Box\alpha'_1, \neg l \in \Box\alpha''_1$.

Now we can deduce the formulas not containing *res* and the occurrences of considered complementary literals.

a) We derive $\alpha'_0 \vee \alpha''_0 \vee \Box\alpha'_1 \vee \ldots \vee \Box\alpha'_m \vee \Box\alpha''_1 \vee \ldots \vee \Box\alpha''_n \vee \beta' \vee \beta''$ from $res(l \vee \alpha'_0 \vee \Box\alpha'_1 \vee \ldots \vee \Box\alpha'_m \vee \beta', \neg l \vee \alpha''_0 \vee \Box\alpha''_1 \vee \ldots \vee \Box\alpha''_n \vee \beta'')$.
b) We derive $\alpha'_0 \vee \alpha''_0 \vee \Box\alpha'_1 \vee \ldots \vee \Box\alpha'_m \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta' \vee \beta''$ from $res(l \vee \alpha'_0 \vee \Box\alpha'_1 \vee \ldots \vee \Box\alpha'_m \vee \beta', \alpha'' \vee \Box(\neg l \vee \alpha''_0) \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta'')$.
c) We derive $\alpha' \vee \alpha'' \vee \Box(\alpha'_0 \vee \alpha''_0) \vee \Box\alpha'_2 \vee \ldots \vee \Box\alpha'_m \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta' \vee \beta''$ and $\alpha' \vee \alpha'' \vee \alpha'_0 \vee \alpha''_0 \vee \Box\alpha'_2 \vee \ldots \vee \Box\alpha'_m \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta' \vee \beta''$ from $res(\alpha' \vee \Box(l \vee \alpha'_0) \vee \Box\alpha'_2 \vee \ldots \vee \Box\alpha'_m \vee \beta', \alpha'' \vee \Box(\neg l \vee \alpha''_0) \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta'')$. One can see that the obtained clauses are exactly of the form (3).

Case 2. One member of a complementary pair has the form $\Diamond l$ or $\Diamond\ Boxl$. Now we have the following subcases: a) $\Diamond l \in \alpha', \Box\neg l \in \alpha''$, b) $\Box l \in \alpha', \Box\neg l \in \Box\alpha''_1$, c) $\Diamond l \in \alpha'_1, \neg l \in \Box\alpha''_1$, d) $\Diamond l \in \Box\alpha'_1, \Box\neg l \in \alpha''$, e) $\Diamond l \in \Box\alpha'_1, \neg l \in \Box\alpha''_1$, f) $\Diamond l \in \Box\alpha'_1, \Box\neg l \in \Box\alpha''_1$, g) $\Diamond\Box l \in \beta', \Box\neg l \in \alpha''$, h) $\Diamond\Box l \in \beta', \neg l \in \Box\alpha''_1$, i) $\Diamond\Box l \in \beta', \Box\neg l \in \Box\alpha''_1$, j) $\Diamond\Box \in \beta', \Diamond\neg l \in \Box\alpha''_1$.
We shall consider only the subcase *e*.

e) We derive $\alpha' \vee \alpha'' \vee \alpha'_0 \vee \Diamond\alpha''_0 \vee \Box\alpha'_2 \vee \ldots \vee \Box\alpha'_m \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta' \vee \beta''$ from $res(\alpha' \vee \Box(\Diamond l \vee \alpha'_0) \vee \Box\alpha'_2 \vee \ldots \vee \Box\alpha'_m \vee \beta', \alpha'' \vee \Box(\neg l \vee \alpha''_0) \vee \Box\alpha''_2 \vee \ldots \vee \Box\alpha''_n \vee \beta'')$. If α''_0 is a modal clause, then we have a required form. If α''_0 is a disjunction of modal clauses, then we bring $\Diamond$ in the brackets using statement that $\Gamma, \Diamond(F \vee G) \vdash$ is derivable in S4 if and only if $\Gamma, \Diamond F \vee \Diamond G \vdash$ is derivable in S4. In addition, we change the formulas of the form $\Diamond\Diamond F$ by $\Diamond F$. Aftewards we obtain a formula of the form (3).

Case 3. A complementary pair has the form $\Box l, \ \neg l$.

Case 4. A complementary pair has the form $\Box l, \ \Box\neg l$.

We consider last both the cases in a way similar to case 1. We obtain a formula of the form (3). We apply the rule of factorization to the obtained formula as long as it possible. That is, we obtain a formula, for which the rules of factorization and duplication are not more applicable. Theorem is proved.

EXAMPLE. *Let us consider the formula* $\neg(\Box\forall x A(x) \to \forall x \Box A(x))$. *We investigate the correponding set* $S = \{\Box\forall x A(x), \exists x \Diamond\neg A(x)\}$. *The formulas are of the form (1). We shall now show that S is a contradictory set in S4. We skolemize the formulas.* $S' = \{\Box\forall x A(x), \Diamond\neg A(b)\}$. $H_0^0 = \{a, b\}$. $H_j^i = H_0^0$ *for* $i = 0, 1;\ j \geqslant 0$. $S_j' = \{\Box A(a), \Box A(b), \Diamond\neg A(b)\}$ *for* $j \geqslant 0$. *We have the following derivation*

$$\cfrac{\cfrac{\cfrac{\cfrac{\Box A(b),\quad \Diamond\neg A(b)}{res(\Box A(b), \Diamond\neg A(b))}}{\Diamond res(A(b), \neg A(b))}}{\Diamond \bot}}{\bot}$$

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REZIUMĖ

S. Norgėla. Rezoliucijų metodas vienai modalumo logikos S4 klasei

Nagrinėjamos kvantorinės modalumo logikos formulės pavidalo

$$\forall^*\Box^i\forall^*\exists^*(A_1 \vee \ldots \vee A_s \vee \Box B_1 \vee \ldots \vee \Box B_u \vee \Diamond C_1 \vee \ldots \vee \Diamond C_v).$$

Čia A_i, B_i, C_i yra klasikinės logikos literos. Tokio pavidalo formulėms aprašytas rezoliucijų metodas modalumo logikoje S4. Be to, aprašytas išvedamų disjunktų pavidalas.