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Runge–Kutta-type methods for solving two-dimensional stochastic differential equations

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Introduction. Runge–Kutta-type methods for SDEs

We consider a two-dimensional Stratonovich stochastic differential equation of the form

$$X_i(t) = x_i + \int_0^t f_i(X(t))\,\mathrm{d}t + \sum_{j=1}^{2} \int_0^t g_{ij}(X(t))\,\mathrm{d}B_j(t), \quad t \in [0, T],\ i = 1, 2,$$

or, in matrix notation,

$$X(t) = x + \int_0^t f(X(t))\,\mathrm{d}t + \int_0^t g(X(t))\,\mathrm{d}B(t), \quad t \in [0, T], \tag{1}$$

where $B = (B_1, B_2)$ is a two-dimensional Brownian motion, and $f_i\colon \mathbb{R}^2 \to \mathbb{R}$, $g_{ij}\colon \mathbb{R}^2 \to \mathbb{R}$, $i, j = 1, 2$.

A family of stochastic processes $\{X^h,\ h > 0\}$ is called a second-order weak approximation of the solution X (in the time interval $[0, T]$) if, for all $t \in [0, T]$,

$$E\varphi(X^h(t)) - E\varphi(X(t)) = \mathrm{O}(h^2), \quad h \to 0.$$

for a "rather wide" class of functions $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ (see [1], [2]).

We consider weak approximations of the form

$$X^h(0) = x, \quad X^h(t_{k+1}) = A\big(X^h(t_k), h, \Delta B_k\big), \quad k = 0, 1, \dots, N,$$

where $t_k = t_k(h) = kh$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and the function $A\colon \mathbb{R}^2 \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is such that

$$A(x, 0, 0) \equiv x.$$

(We set $X^h(t) := X^h(t_k)$, $t \in [t_k, t_{k+1})$.)

Similarly to [2], [4], we define a Runge–Kutta-type method for a two-dimensional stochastic differential equation by

$$A(x, s, b) := x + \sum_{i=0}^{3} q_i F_i s + \sum_{i=0}^{3} r_i G_i b, \tag{2}$$

where

$$\begin{aligned}
F_0 &= f(x), \quad F_1 = f(x + \alpha_{10}F_0 s + \beta_{10}G_0 b),\\
G_0 &= g(x + \alpha_{00}F_0 s), \quad G_1 = g(x + \alpha_{10}F_0 s + \beta_{10}G_0 b),\\
F_2 &= f\big(x + (\alpha_{20}F_0 + \alpha_{21}F_1)s + (\beta_{20}G_0 + \beta_{21}G_1)b\big),\\
G_2 &= g\big(x + (\alpha_{20}F_0 + \alpha_{21}F_1)s + (\beta_{20}G_0 + \beta_{21}G_1)b\big),\\
F_3 &= f\big(x + (\alpha_{30}F_0 + \alpha_{31}F_1 + \alpha_{32}F_2)s + (\beta_{30}G_0 + \beta_{31}G_1 + \beta_{32}G_2)b\big),\\
G_3 &= g\big(x + (\alpha_{30}F_0 + \alpha_{31}F_1 + \alpha_{32}F_2)s + (\beta_{30}G_0 + \beta_{31}G_1 + \beta_{32}G_2)b\big),
\end{aligned} \tag{3}$$

and

$$F_i = \begin{pmatrix} F_i^1 \\ F_i^2 \end{pmatrix}, \quad G_i = \begin{pmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{pmatrix}. \tag{4}$$

It seems (cf. [3]) that, for n-dimensional Stratonovich SDEs, second-order Runge–Kutta methods of the form (3) do not exist. However, one can expect to construct such methods for some simpler classes of diffusion coefficients g. In this short note, we consider the cases where

$$g = \begin{pmatrix} g_{11}(x_1, x_2) & 0 \\ 0 & 0 \end{pmatrix} \tag{5}$$

or

$$g = \begin{pmatrix} g_{11}(x_1) & 0 \\ 0 & g_{22}(x_2) \end{pmatrix}. \tag{6}$$

In this paper, we construct a second-order Runge–Kutta approximation for both cases.

Second-order conditions for weak Runge–Kutta approximations

In [3], sufficient conditions for the second-order accuracy of a weak approximation are obtained. Substituting these conditions into (2) and using MAPLE for calculations, in the cases (5) and (6), we obtain the following 16 equations for the parameters q_i, r_i, α_{ij}, β_{ij} of a Runge–Kutta type approximation:

$$q_0 + q_1 + q_2 + q_3 = 1, \tag{7}$$

$$r_1\beta_{10} + r_2(\beta_{20} + \beta_{21}) + r_3(\beta_{30} + \beta_{31} + \beta_{32}) = \frac{1}{2}, \tag{8}$$

$$(r_0 + r_1 + r_2 + r_3)^2 = 1, \tag{9}$$

$$q_1\alpha_1 + q_2\alpha_2 + q_3\alpha_3 = \frac{1}{2}, \tag{10}$$

$$q_1\beta_1^2 + q_2\beta_2^2 + q_3\beta_3^2 = \frac{1}{2}, \tag{11}$$

$$r_1\alpha_1\beta_1 + r_2\alpha_2\beta_2 + r_3\alpha_3\beta_3 = \frac{1}{4}, \tag{12}$$

$$r_1\beta_1^3 + r_2\beta_2^3 + r_3\beta_3^3 = \frac{1}{4}, \tag{13}$$

$$r_2\alpha_{21}\beta_1 + r_3\alpha_{31}\beta_1 + r_3\alpha_{32}\beta_2 = 0, \tag{14}$$

$$q_2\beta_{21}\beta_1 + q_3\beta_{31}\beta_1 + q_3\beta_{32}\beta_2 = \frac{1}{4}, \tag{15}$$

$$r_3\beta_{32}\beta_{21}\beta_1 = \frac{1}{24}, \tag{16}$$

$$r_0\alpha_0 + r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3 = \frac{1}{2}, \tag{17}$$

$$r_1\beta_{10}\alpha_0 + r_2\beta_{20}\alpha_0 + r_2\beta_{21}\alpha_1 + r_3\beta_{30}\alpha_0 + r_3\beta_{31}\alpha_1 + r_3\beta_{32}\alpha_2 = \frac{1}{4}, \tag{18}$$

$$\begin{aligned}&2r_3\beta_{32}^2\beta_{21} + 2r_2\beta_{21}^2\beta_{10} + r_3\beta_{32}\beta_{20}^2 + r_2\beta_{21}\beta_{10}^2 + r_3\beta_{31}\beta_{10}^2 + 2r_3\beta_{32}^2\beta_{20}\\ &\quad + r_3\beta_{32}\beta_{21}^2 + 2r_2\beta_{21}\beta_{10}\beta_{20} + 2r_3\beta_{32}\beta_{20}\beta_{30} + 2r_3\beta_{31}^2\beta_{10} + 2r_3\beta_{32}\beta_{21}\beta_{31}\\ &\quad + 2r_3\beta_{32}\beta_{21}\beta_{30} + 2r_3\beta_{32}\beta_{20}\beta_{31} + 2r_3\beta_{32}\beta_{10}\beta_{31} + 2r_3\beta_{31}\beta_{10}\beta_{30}\\ &\quad + 2r_3\beta_{32}\beta_{20}\beta_{21} = \frac{1}{3},\end{aligned} \tag{19}$$

$$q_1\beta_1 + q_2\beta_2 + q_3\beta_3 = \frac{1}{2}, \tag{20}$$

$$r_1\beta_1^2 + r_2\beta_2^2 + r_3\beta_3^2 = \frac{1}{3}, \tag{21}$$

$$r_2\beta_{21}\beta_1 + r_3\beta_{31}\beta_1 + r_3\beta_{32}\beta_2 = \frac{1}{6}. \tag{22}$$

Solving this equation system (again using MAPLE), we have obtained several solutions of Eqs. (7)–(22). It is convenient to express them in the form of the Butcher-type array

$$\begin{array}{c|cccc} \alpha_0 & \alpha_{00} & & & \\ \alpha_1 & \alpha_{10} & & & \\ \alpha_2 & \alpha_{20} & \alpha_{21} & & \\ \alpha_3 & \alpha_{30} & \alpha_{31} & \alpha_{32} & \\ \hline & q_0 & q_1 & q_2 & q_3 \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \beta_1 & \beta_{10} & & & \\ \beta_2 & \beta_{20} & \beta_{21} & & \\ \beta_3 & \beta_{30} & \beta_{31} & \beta_{32} & \\ \hline & r_0 & r_1 & r_2 & r_3 \end{array}$$

We have chosen the nicest one with "many" zeros:

$$
\begin{array}{c|cccc}
\frac{2}{3} & \frac{2}{3} & & & \\
\frac{2}{3} & \frac{2}{3} & & & \\
\frac{1}{6} & \frac{1}{6} & 0 & & \\
\frac{5}{12} & \frac{5}{12} & 0 & 0 & \\
\hline
 & -\frac{1}{2} & \frac{1}{2} & 1 & 0
\end{array}
\qquad
\begin{array}{c|cccc}
0 & & & & \\
1 & & & & \\
0 & -\frac{1}{4} & \frac{1}{4} & & \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & \\
\hline
 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{2}{3}
\end{array}
$$

Simulation example

We illustrate our approximation by a Hamilton-type equation system

$$
\begin{cases}
\mathrm{d}V(t) = \big(f(X(t)) - \alpha V(t)\big)\,\mathrm{d}t + g\big(V(t), X(t)\big)\,\mathrm{d}B_t, \\
\mathrm{d}X(t) = V(t)\,\mathrm{d}t,
\end{cases}
$$

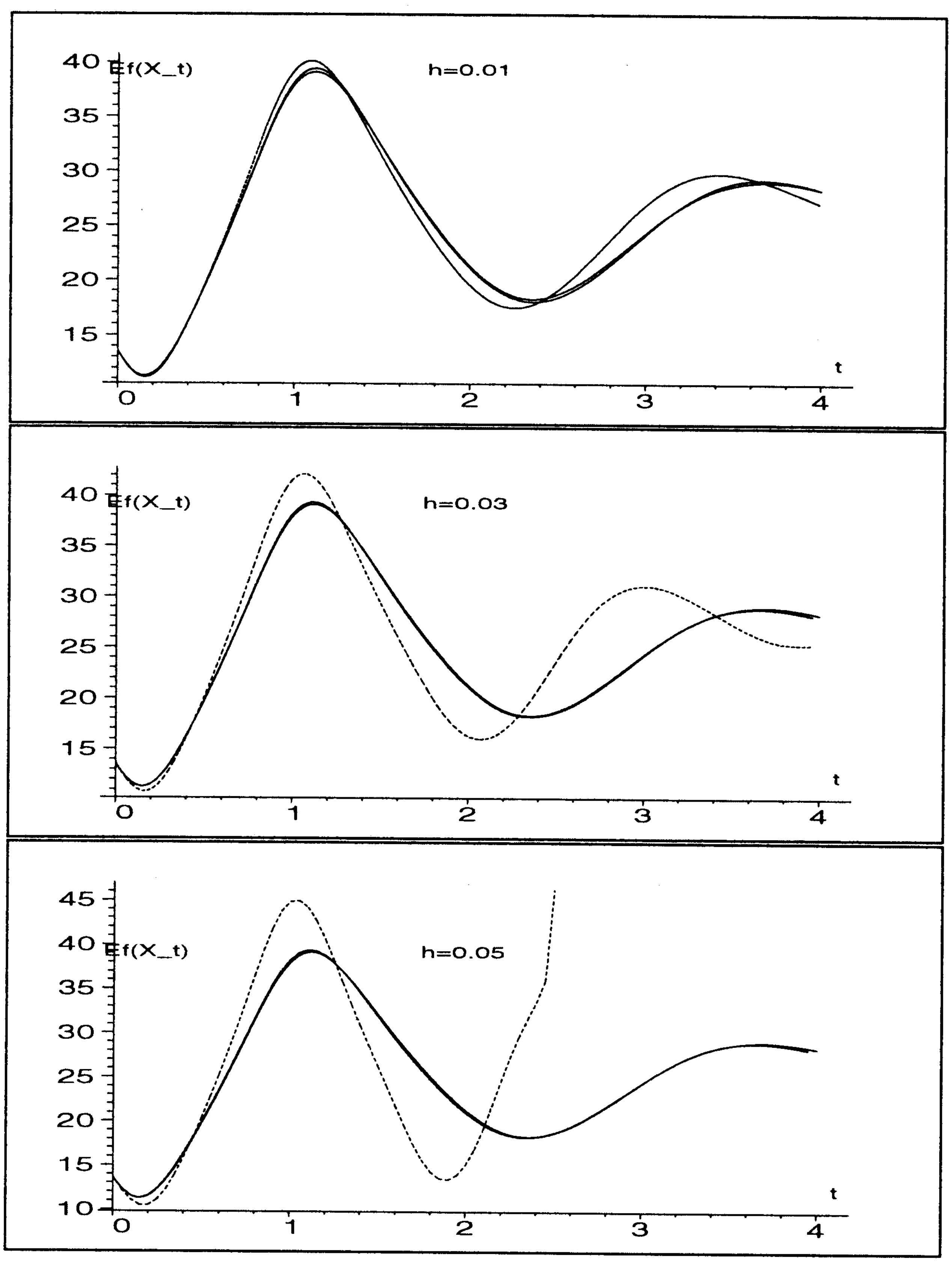

Fig. 1. Solid line: $Ef(X_t)$ with exact solution X.
Solid polygonal black line: $Ef(X_t^{RK})$ with the Runge–Kutta type approximation X^{RK}.
Dashed polygonal black line: $Ef(X_t^{EM})$ with the Euler–Maruyama approximation X^{EM}.

or, in matrix notation,

$$\begin{pmatrix} V(t) \\ X(t) \end{pmatrix} = \begin{pmatrix} V_0 \\ X_0 \end{pmatrix} + \int_0^t \begin{pmatrix} (f(X(t)) - \alpha V(t)) \\ V(t) \end{pmatrix} \mathrm{d}t$$
$$+ \int_0^t \begin{pmatrix} g_{11}(V(t), X(t)) & 0 \\ 0 & 0 \end{pmatrix} \mathrm{d} \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix},$$

Its solution $(X(t), V(t))$, $t \geqslant 0$, describes the position $X(t)$ and velocity $V(t)$ of a particle moving under the influence of the potential H, of viscosity (or friction), and of random perturbations of velocity. Here $\alpha > 0$ is a constant, and B is a standard Brownian motion. We test our approximation with $f(x) = -4x^3 + 2x$, $\alpha = 0.5$, $g_{11}(v, x) = \sqrt{v^2 + x^2 + 1}$, and $\varphi(V, X) = 10 + (X - 4)^2 + V$. Here $f(x) = -U'(x)$ corresponds to the Duffing potential $U(x) = x^4 - x^2$ popular in mathematical physics.

Comparing the simulations obtained with those obtained using the (first-order) Euler–Maruyama weak approximation, we see (Fig. 1) that the Runge–Kutta approximation behaves significantly better (especially, for "large" values of h).

References

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REZIUMĖ

J. Navikas. Dvimačių stochastinių diferencialinių lygčių Rungės–Kuto sprendimo metodai

Šiame straipsnelyje nagrinėjamos silpnosios antrosios eilės Rungės–Kuto tipo aproksimacijos dvimatėms stochastinėms diferencialinėms lygtims su paprasta difuzijos matrica. Žinoma, kad bendru atveju, tokios Rungės–Kuto tipo aproksimacijos neegzistuoja. Tačiau iš kitos pusės daugiamatės stochastinės lygtys su viena koordinate (funkcija) naudojamos realių procesų simuliacijai (dažniausiai fizikoje). Todėl prasminga nagrinėti šiuos atvejus. Taigi šiame straipsnelyje parodyta, kad Rungės–Kuto aproksimacija egzistuoja, ir pateiktas konkretus pavyzdys.