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On noncommutative Hilbert rings

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1. Introduction

The Hilbert algebraic Nullstellensatz is traditionally stated in several different ways. We recall two best known algebraic forms. One, sometimes called a Weak Nullstellensatz, says that if $\mathfrak{M}$ is a maximal ideal in a polynomial ring $k[X_1, \dots, X_n]$ over a field k, then the field $k[X_1, \dots .X_n]/\mathfrak{M}$ is a finite-dimentional extension of k for all $n \in \mathbb{N}$. The second statement is that in the ring $k[X_1, \dots, X_n]$ every prime ideal is an intersection of maximal ideals or, in terms of radicals, in each factor ring of $k[X_1, \dots, X_n]$ the nilradical coincides with the Jacobson radical. Each of these results easily implies the classical Hilbert theorem about zeros in polynomial rings. There is a very natural question: which rings R have the property that for every $n \in \mathbb{N}$ and every maximal ideal $\mathfrak{M} \subset R[X_1, \dots, X_n]$ the factor ring $R[X_1, \dots, X_n]/\mathfrak{M}$ is finitely generated as a canonical R-module. When R is commutative, it has this property exactly when every prime ideal in R is an intersection of maximal ideals. See [6] and [10]. This result gave the motivation of the following well known definition.

A commutative ring R is called a *Hilbert ring*, also *Jacobson* or *Jacobson-Hilbert ring* (see, e.g., [3, Chapter 4]) if every prime ideal of R is the intersection of maximal ideals. This is obviously equivalent to require that in each factor ring of R the nilradical coincides with the Jacobson radical. Evidently, the class of commutative Hilbert rings is closed under forming factor rings. The interest in this class of rings is based on the following characterizations.

For a commutative ring R the following are equivalent:

(a) *R is a Hilbert ring;*
(b) *for each maximal ideal $\mathfrak{M} \subset R[X]$, the intersection $\mathfrak{M} \cap R$ is a maximal ideal in R;*
(c) *every maximal ideal of $R[X]$ contains a monic polynomial;*
(d) *every prime ideal $\mathfrak{p} \subset R$ is maximal or is an intersection of prime ideals properly containing $\mathfrak{p}$;*
(e) *the polynomial ring $R[X]$ is a Hilbert ring.*

So the class of commutative Hilbert rings is closed under forming finite polynomial rings. For an integral extension $R \subseteq S$, the ring S is Hilbert if and only if R is a Hilbert ring. The main interest in Hilbert rings in commutative algebra and algebraic geometry is their relation with Hilbert's Nullstellensatz. In this article we give the answer to the earlier question. Which, not necessarily commutative rings, have the property that for every maximal (two-sided) ideal $\mathfrak{M} \subset R[X_1, \ldots, X_n]$ the factor ring $R[X_1, \ldots, X_n]/\mathfrak{M}$ is finitely generated as a canonical R-module for all $n \in \mathbb{N}$.

Having in mind this question we extend the notion of commutative Hilbert rings to any noncommutative rings by restricting the requirements to a special class of prime ideals, the *strongly prime ideals.* These are related to strongly prime rings which can be characterized by the fact that their central closures (in the sense of Martindale) are simple rings. We give a characterization of Hilbert rings in terms of a contraction property in the rings of polynomials, and the class of Hilbert rings is the largest class of rings having this property. Furthermore, we characterise Hilbert rings in terms of monic polynomials, show that the class of Hilbert rings is closed under finite polynomial and integral extensions, and obtain a natural symmetric form of a general Hilbert's Nullstellensatz for Hilbert rings. Note, that we do not use the commutative Nullstellensatz in the proofs and obtain it as a corrollary.

All rings in this paper are associative with identity element which should be preserved by ring homomorphisms. By an *ideal* of the ring we shall understand a two-sided ideal. $A \subset B$ means that A is proper subset of B. Throughout $R[X_1, \ldots, X_n]$ will denote the polynomial ring with n commuting indeterminates, which also commute with the elements from the ring R.

2. Strongly prime ideals

Let R be a prime ring, $Q(R)$ its central closure and $F(R)$ the extended centroid of the ring R, which is the centre of $Q(R)$. As shown in [15, Section 32], $Q(R)$ may be understood as self-injective hull of R as $R \otimes_{\mathbb{Z}} R^o$-module. Any element $\varphi \in F(R)$ may be represented as an R-bimodule homomorphism $\varphi\colon I \to R$ for some nonzero ideal $I \subseteq R$, called a *domain of definition* of φ. Evidently $\varphi I \subseteq R$ in $Q(R)$.

Lemma 2.1. *Let R be a prime ring, $\varphi_1, \ldots, \varphi_m$ a finite subset of elements from the extended centroid $F(R)$, and $\mathcal{P} \subset R$ a prime ideal. Assume that every element φ_k has a domain of definition which is not contained in $\mathcal{P}$. Then the extension $\mathcal{P}^e$ of the ideal $\mathcal{P}$ in $R[\varphi_1, \ldots, \varphi_m] \subseteq Q(R)$ is a proper ideal.*

Proof. Let φ_k, $1 \leqslant k \leqslant m$, be defined on the ideal $I_k \not\subseteq \mathcal{P}$. If $\mathcal{P}^e$ is not a proper ideal in $R[\varphi_1, \ldots, \varphi_m]$ we have an expression $1 = p_1 u_1 + \cdots + p_n u_n \in \mathcal{P}^e$ with some $p_k \in \mathcal{P}$ and the u_k being products of the given elements from the extended centroid. So all these functions u_k are defined on some nonzero finite product of the ideals I_j, which is not contained in the prime ideal $\mathcal{P}$, a contradiction.

Recall that a prime ring R is *strongly prime* if its central closure $Q(R)$ is a simple ring. Various characterizations of strongly prime rings are given, e.g., in [15, 35.6] and [8, Theorem 2.1]. For any strongly prime ring R the central closure $Q(R)$ is left and right flat as R-module and there is a canonical isomorphism $Q(R) \otimes_R Q(R) \cong Q(R)$, see

Theorem 2.8 in [8]. This means that the canonical inclusion $R \hookrightarrow Q(R)$ is a left and right ring epimorphism and the family of the left ideals $\mathcal{F} = \{L \subseteq R \mid QL = Q\}$ is a Gabriel filter giving a perfect localization in R for which $R_{\mathcal{F}}$ is canonically isomorphic to $Q(R)$.

Let $\phi: R \to S$ be a ring homomorphism. Then S becomes a canonical R-bimodule and we write rs and sr instead of $\phi(r)s$ and $s\phi(r)$ for $r \in R$, $s \in S$. Let $Z_S(R) = \{x \in S \mid rx = xr, \forall r \in R\}$ be the set of R-centralizing elements of the ring S.

We call ϕ a *centred homomorphism* and S a centred extension of R via ϕ, provided $S = RZ_S(R)$. This means that $s = \sum_k r_k x_k$ for each element $s \in S$ with some $r_k \in R$ and $x_k \in Z_S(R)$. If $Z_S(R)$ is commutative then centred extensions are called *central extensions.* Rings and their centred homomorphisms form a category, known as *Procesi category.* A centred extension $R \subseteq S$ is called a *liberal extension* if S is finitely generated as a canonical R-module.

An ideal $\mathfrak{p} \subset R$ is called *strongly prime* if the factor ring $R/\mathfrak{p}$ is a strongly prime ring. The following lemma recalls some crucial properties.

Lemma 2.2. *Let $\phi: R \to S$ be a centred homomorphism.*

(1) *Assume S to be a simple ring. Then:*

  (i) *The kernel of ϕ is a strongly prime ideal in R.*
  (ii) *If ϕ is injective, there is a unique extending ring homomorphism $Q(R) \to S$ which maps the center of $Q(R)$ into the center of S.*

(2) *Assume $R \subseteq S$ to be a liberal extension. Then for any prime ideal $\mathfrak{p}$ in R there is a prime ideal ideal $\mathfrak{P}$ in S with $\mathfrak{P} \cap R = \mathfrak{p}$ (lying over).*

Proof. (1) (i) follows from the characterisations of strongly prime rings (see [8, Theorem 2.1]). (ii) follows from Amitsur [2, Theorem 18].

(2) This is shown in [12, Theorem 4.1].

The intersection of all strongly prime ideals of the ring R is called the *strongly prime radical* and we denote it by $SP(R)$. We recall, that the intersection of all maximal (two-sided) ideals of the ring R is called Brown-McCoy radical of the ring, which we denote $BMc(R)$. These two radicals are closely related.

Theorem 2.3. *In any nonzero ring R,*

$$SP(R) = \bigcap_{n \geqslant 1} (R \cap BMc(R[X_1, \ldots, X_n])).$$

Proof. If $a \in R$ does not belong to some maximal ideal $\mathfrak{M} \subset R[X_1, \ldots. X_n]$, then $a \notin \mathfrak{M} \cap R = \mathfrak{p}$. Clearly $R \to R[X_1, \ldots, X_n]/\mathfrak{M}$ is a centred extension and, by 2.2, $\mathfrak{p} \subset R$ is a strongly prime ideal. So $a \notin SP(R)$. If $a \notin \mathfrak{p}$ for some strongly prime ideal

$\mathfrak{p} \subset R$, then in the central closure $Q(R/\mathfrak{p})$, which is a simple ring by definition of the strongly prime ideals, and we have an expression

$$a_1\varphi_1 + \cdots + a_n\varphi_n = 1,$$

where $a_1, \dots, a_n \in (a) = RaR$, and $\varphi_1, \dots, \varphi_n$ are from the extended centroid of the ring $R/\mathfrak{p}$, which is the centre of $Q(R/\mathfrak{p})$. So, sending X_k to φ_k for $1 \leqslant k \leqslant n$, we obtain the centred homomorphism $\phi: R[X_1, \dots, X_n] \to Q(R/\mathfrak{p})$. Evidently, the polynomial $a_1X_1 + \cdots + a_nX_n - 1$ is in the $ker\phi$. If $\mathfrak{M} \subset R[X_1, \dots, X_n]$ is a maximal ideal containing $ker\phi$, then $a \notin \mathfrak{M}$, and $a \notin BMc(R[X_1, \dots, X_n])$. So we have proved both inclusions.

In terms of elements, $a \in SP(R)$ if and only if for any $a_1, \dots, a_n \in (a)$, the ideal in $R[X_1, \dots, X_n]$, generated by the polynomial $a_1X_1 + \cdots + a_nX_n - 1$ contains 1_R (see [8, Theorem 3.2]).

Theorem 2.4. *Let $\mathfrak{M} \subset R[X_1, \dots, X_n]$ be a maximal ideal and $\mathfrak{p} = \mathfrak{M} \cap R$. Then the following are equivalent:*

(a) *$\mathfrak{p}$ is the maximal ideal in R;*
(b) *there exists a monic polynomial $f(T) \in R[T]$ such that $f(X_k) \in \mathfrak{M}$ for all $1 \leqslant k \leqslant n$;*
(c) *$R/\mathfrak{p} \subseteq R[X_1, \dots, X_n]/\mathfrak{M}$ is a liberal extension.*

Proof. $(a) \Rightarrow (b)$ Induction. Denote

$$\mathcal{A} = R/\mathfrak{p} \hookrightarrow R[X_1, \dots, X_n]/\mathfrak{M} = \mathcal{B} = \mathcal{A}[x_1, \dots, x_n]$$

where $x_1, \dots, x_n$ are the images of $X_1, \dots, X_n$ respectively, and these elements are from the centre of $\mathcal{B}$.

Case $n = 1$ is trivial. Let $n \in \mathbb{N}$. If all $x_1, \dots, x_{n+1}$ are algebraic over $\mathcal{A}$ then statement (b) is evidently true. Indeed, then all x_k satisfy the monic polynomial $g_k(T) \in \mathcal{A}[T]$. Lifting the product $g_1 \cdots g_n$ to a monic polynomial $f(T) \in R[T]$ we obtain that $f(X_k) \in \mathfrak{M}$, $1 \leqslant k \leqslant n$. Let some element, say x_1, be transcendental over $\mathcal{A}$. If the field F is the centre of $\mathcal{A}$, and $\mathcal{S}$ the multiplicative set of nonzero elements of the ring $F[x_1]$ then the central localization $\mathcal{A}[x_1]_{\mathcal{S}} \subseteq \mathcal{B}$ is the simple ring canonically isomorphic to $\mathcal{A}(x_1) = \mathcal{A} \otimes_F F[x_1]$. By inductive hypothesis, elements $x_2, \dots, x_{n+1}$ satisfy a monic polynomial over $\mathcal{A}[x_1]_{\mathcal{S}}$, so these elements satisfy a monic polynomial over $\mathcal{A}[x_1]_h$, where $h = h(x_1) \in F[x_1]$ is non constant central polynomial. So $\mathcal{B}$ is a liberal extension of $\mathcal{A}[x_1]_h$ and by Lemma 2.2(2), $\mathcal{A}[x_1]_h$ is a simple ring. But this is impossible. Indeed, the ideal in $F[x_1]$ generated by nonconstant polynomial $h - 1$ is a nonzero proper ideal which does not contain h. So maximal ideal of $F[x_1]$ containing $h - 1$ does not contain h^n, $n \in \mathbb{N}$ and its extension in $\mathcal{A}[x_1]$ is also a proper nonzero ideal which does not contain h^n, $n \in \mathbb{N}$ because $\mathcal{A}$ is a simple ring. A contradiction with an assumption that x_1 is transcendental over $\mathcal{A}$.

Evidently, (c) follows from (b).

$(c) \Rightarrow (a)$ The simple ring $\mathcal{S}$ is a liberal extension of the ring $\mathcal{A}$. By Lemma 2.2(2), this implies that $\mathcal{A}$ is also a simple ring, so the ideal $\mathfrak{p} = \mathfrak{M} \cap R$ is maximal.

Theorem 2.5. *Let $\mathfrak{M} \subset R[X_1, \ldots, X_n]$ be a maximal ideal and $\mathfrak{p} = \mathfrak{M} \cap R$. If $\mathfrak{p}$ is not maximal in R then the intersection of all nonzero prime ideals in $R/\mathfrak{p}$ is not zero.*

Proof. Using Lemma 2.2(1)(ii) and notations above we have

$$\mathcal{A} = R/\mathfrak{p} \subset Q(\mathcal{A}) \hookrightarrow R[X_1, \ldots, X_n]/\mathfrak{M} = \mathcal{B} = \mathcal{A}[x_1, \ldots, x_n] = Q(\mathcal{A})[x_1, \ldots, x_n]$$

where $Q(\mathcal{A}$ is the central closure of the ring $\mathcal{A}$. By Theorem 2.4(b) all elements x_k satisfy the monic polynomial over the simple ring $Q(\mathcal{A})$, so they satisfy the monic polynomial over $\mathcal{A}[\varphi_1, \ldots, \varphi_m] \subseteq Q(\mathcal{A})$, where elements $\varphi_1, \ldots, \varphi_m$ are from the extended centroid of the ring $\mathcal{A}$. By Theorem 2.4(c), $\mathcal{A}[\varphi_1, \ldots, \varphi_m]$ is the simple ring and coincides with $Q(\mathcal{A})$. So by Lemma 2.1 and Lemma 2.2(2) the intersection of all nonzero prime ideals of $\mathcal{A}$ is nonzero.

3. Noncommutative Hilbert rings

We call a ring R a *Hilbert ring* if every strongly prime ideal is the intersection of maximal ideals. This is obviously equivalent to require that in each factor ring of R the strongly prime radical coincides with the Brown-McCoy radical. From this it is clear that any factor ring of a Hilbert ring is again Hilbert. Since in commutative rings strongly prime ideals are precisely prime ideals, this extends the notion of commutative Hilbert rings. Moreover, the class of Hilbert rings contains the Brown-McCoy rings considered in [14] and also the Jacobson PI rings considered in [1] and [11].

Since, by Theorem 2.5 in [8], strongly prime ideals are preserved under Morita equivalence, it follows that the property to be a Hilbert ring is preserved under Morita equivalence.

The importance of this notion is based on the following characterizations.

Theorem 3.1. *For any ring R the following are equivalent:*

(a) *R is a Hilbert ring;*
(b) *for each $n \in \mathbb{N}$ and any maximal ideal $\mathfrak{M}$ of $R[X_1, \ldots, X_n]$, the contraction $\mathfrak{M} \cap R$ is a maximal ideal of R;*
(c) *for each $n \in \mathbb{N}$ and every maximal ideal $\mathfrak{M}$ of $R[X_1 \ldots, X_n]$, there exists a monic polynomial $f(T) \in R[T]$, such that $f(X_k) \in \mathfrak{M}$, for $1 \leqslant k \leqslant n$;*
(d) *for each $n \in \mathbb{N}$ and every maximal ideal $\mathfrak{M} \subset R[X_1, \ldots, X_n]$, the extension $R/\mathfrak{M} \cap R \hookrightarrow R[X_1, \ldots, X_n]/\mathfrak{M}$ is liberal;*
(e) *every strongly prime ideal $\mathfrak{p} \subset R$ is maximal or is the intersection of prime ideals properly containing $\mathfrak{p}$;*

(f) *the polynomial ring $R[X]$ is a Hilbert ring.*

Proof. Evidently, by Theorems 2.5 and 2.4 $(a) \Rightarrow (e) \Rightarrow (b)$.

$(b) \Rightarrow (a)$ Let $\mathfrak{M}_\alpha$, $\alpha \in \mathcal{I}_n$ be the family of maximal ideals of the polynomial ring $R[X_1, \ldots, X_n]$. We have:

$$R \cap BMc(R[X_1, \ldots, X_n]) = R \cap (\bigcap_\alpha \mathfrak{M}_\alpha) = \bigcap_\alpha (R \cap \mathfrak{M}_\alpha) = BMc(R),$$

because all ideals $R \cap \mathfrak{M}$ are maximal in R, and each maximal ideal of R can be obtained in this way. So, by Theorem 2.3, $SP(R) = BMc(R)$. Since (b) also holds for each factor ring of the ring R, so does (a) and R is a Hilbert ring.

The equivalence $(b) \Leftrightarrow (c) \Leftrightarrow (d)$ follows easily from Theorem 2.4.

$(a) \Rightarrow (f)$ Since we already know that (a) and (d) are equivalent, it suffices to show that $R[X]$ satisfies (d). Take a maximal ideal $\mathfrak{M} \subset R[X, X_1, \ldots, X_n]$. By (d), an extension $R/\mathfrak{M} \cap R \hookrightarrow R[X, X_1, \ldots, X_n]/\mathfrak{M}$ is liberal, so an extexsion $R[X]/\mathfrak{M} \cap R[X] \hookrightarrow R[X, X_1, \ldots, X_n]/\mathfrak{M}$ is also liberal. So $R[X]$ satisfies (d) and is a Hilbert ring. Since factor rings of the Hilbert rings are again Hilbert, we obtain that $(f) \Rightarrow (a)$.

So we obtain the following corollary which may be considered as a *general symmetric noncommutative form of Hilbert's Nullstellensatz*, see [3, Theorem 4.19].

COROLLARY 3.2. If R is a Hilbert ring, then the ring $R[X_1, \ldots, X_n]$ is also a Hilbert ring. In this case, for each maximal ideal $\mathfrak{M} \subset R[X_1, \ldots, X_n]$, the factor ring $R[X_1, \ldots, X_n]/\mathfrak{M}$ is a central liberal extension of the simple ring $R/\mathfrak{M} \cap R$.

The following definition of integral homomorphisms and integral extensions in Procesi category and their main properties are considered in [7]. A centred ring homomorphism $\phi\colon R \to S$ is called an *integral homomorphism* if every finite subset $\{s_1, \ldots, s_n\} \subseteq S$ is contained in some subring $A \subseteq S$ which is a liberal extension of the ring $\phi(R)$, i.e., A is generated as a canonical R–module by a finite set of R–centralizing elements. In this case S is called an integral extension of R via ϕ. In the commutative case this definition is equivalent to the classical definition of an integral homomorphism. Evidently, liberal extensions of rings are integral, and each integral extension is the inductive limit of liberal extensions. It is clear that an integral extensions of a field are precisely locally finite algebras over this field.The following fundamental properties are shown in the theorems 9, 10 and 11 of [7].

Lemma 3.3. *Let $R \subset S$ be a centred integral extension.*

(1) *For any strongly prime ideal $\mathfrak{p}$ there exists a strongly prime $\mathfrak{q}$ in S such that $\mathfrak{q} \cap R = \mathfrak{p}$ (lying over).*
(2) *If R is simple and S is strongly prime, then S is simple.*

(3) *Consider ideals* $\mathfrak{q} \subseteq A$ *in* S *such that* $\mathfrak{q} \cap R = A \cap R$. *If* $\mathfrak{q}$ *is a strongly prime ideal then* $\mathfrak{q} = A$ *(incomparability).*

This allows us to prove our final result generalising the commutative case.

Theorem 3.4. *Let* $R \subset S$ *be an integral extension of rings. Then* S *is a Hilbert ring if and only if* S *is a Hilbert ring.*

Proof. Let S be a Hilbert ring and $\mathfrak{p} \subset R$ a strongly prime ideal. By 3.3(1), there exists a strongly prime ideal $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$. So $\mathfrak{q}$ is an intersection of maximal ideals in S. Because the contraction of a maximal ideal in S is a maximal ideal in R, we obtain that $\mathfrak{p}$ is the intersection of maximal ideals, so R is a Hilbert ring.

Let R be a Hilbert ring. If $\mathfrak{q} \subset S$ is a strongly prime ideal, then $\mathfrak{q} \cap R$ is a strongly prime ideal in R. Going to factor rings, we reduce the proof to the case of an integral extensions of strongly prime rings. Take a nonzero element $s \in S$. By 3.3(3), an ideal (s) intersects nontrivially with R. So we can find a maximal ideal $\mathfrak{m} \subset R$ which does no tcontain $(s) \cap R$. Now, by 3.3(2), an ideal lying over $\mathfrak{m}$ is a maximal ideal in S which does not contain (s). This means, that the intersection of maximal ideals in S is zero, so S is a Hilbert ring.

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Nekomutatyvieji Hilberto žiedai

A. Kaučikas

Įrodytas nekomutatyvus simetrinis Hilberto Nullstellensatz varijantas. Gautos pagrindinės nekomutatyviųjų Hilberto žiedų charakteristikos