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A note on discrete limit theorems for the Matsumoto zeta-function

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In [1], [2], [3] we obtained discrete limit theorems in the sense of the weak convergence of probability measures in various spaces for the Matsumoto zeta-function $\varphi(s)$, $s = \sigma + it$. The latter function was introduced by K. Matsumoto in [4]. We recall the definition of $\varphi(s)$. Let $g(m)$ and $f(j,m)$ be positive integers, and $a_m^{(j)}$ be complex numbers. Define

$$A_m(X) = \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} X^{f(j,m)}\right),$$

and denote by p_m the mth prime number. Then the function $\varphi(s)$ is defined by the following infinite product

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}). \tag{1}$$

As K. Matsumoto in [4], we suppose that

$$g(m) \leqslant c p_m^{\alpha}, \quad |a_m^{(j)}| \leqslant p_m^{\beta}$$

with some $c > 0$ and some non-negative constants α and β. Then the product in (1) define, for $\sigma > \alpha + \beta + 1$, a holomorphic function without zeros.

Let, for positive integer N,

$$\mu_N(\ldots) = \frac{1}{N+1} \#(0 \leqslant m \leqslant N : \ldots),$$

where instead of dots a condition satisfied by m is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S and let $h > 0$ be a fixed number. Then in [1], [2], [3] the weak convergence of the following probability measures

$$\begin{aligned} &\mu_N(\varphi(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \\ &\mu_N(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(H(D_1)), \\ &\mu_N(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(M(D_2)), \end{aligned} \tag{2}$$

as $N \to \infty$, under some additional conditions on $\varphi(s)$, was investigated. Here $\mathbb{C}$ denotes the complex plane, $H(D_1)$ is the space of analytic on $D_1 = \{s \in \mathbb{C}\colon \sigma > \alpha + \beta + 1\}$ functions, and $M(D_2)$ is the space of meromorphic on $D_2 = \{s \in \mathbb{C}\colon \sigma > \varrho\}$ functions, where $\alpha + \beta + \frac{1}{2} < \varrho < \alpha + \beta + 1$.

In this note we propose a generalization of the mentioned works. Let a positive function $w(t)$ be defined for $t \geqslant 0$, and let

$$U = U(N) = \sum_{m=0}^{N} w(m).$$

Suppose that $\lim_{N\to\infty} U(N) = +\infty$. We put

$$\mu_{N,w}(\ldots) = \frac{1}{U} \sum_{\substack{m=0 \\ \ldots}}^{N} w(m),$$

where instead of dots a condition satisfied by m is to be written. We observe that

$$\mu_{N,1}(\ldots) = \mu_N(\ldots).$$

Then instead of the measures (2) we can consider the weak convergence of probability measures,

$$\begin{aligned}
&\mu_{N,w}(\varphi(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),\\
&\mu_{N,w}(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(H(D_1)),\\
&\mu_{N,w}(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(M(D_2)).
\end{aligned} \tag{3}$$

It turns out that if the function $w(t)$ is non-increasing, then the weak convergence of the probability measures (2) implies that of the measures (3). This follows from the following general theorem. Let $f(t)$ be a S-valued function defined for $t \geqslant 0$, and

$$\begin{aligned}
&P_N(A) = \mu_N(f(mh) \in A), \quad A \in \mathcal{B}(S),\\
&P_{N,w}(A) = \mu_{N,w}(f(mh) \in A), \quad A \in \mathcal{B}(S).
\end{aligned}$$

Theorem. *Suppose that $w(t)$ is a continuous non-increasing function, and that P_N converges weakly to some probability measure P as $N \to \infty$. Then also $P_{N,w}$ converges weakly to P as $N \to \infty$.*

Proof. Since P_N converges weakly to P as $N \to \infty$, we have that

$$\int_S X \, \mathrm{d}P_N \underset{N\to\infty}{\longrightarrow} \int_S X \, \mathrm{d}P \tag{4}$$

for every real bounded continuous function X on S. By the the definition of P_N

$$\int_S X \, \mathrm{d}P_N = \frac{1}{N+1} \sum_{m=0}^{N} X(f(mh)).$$

Therefore, putting

$$\int_S X \, \mathrm{d}P = \kappa_X,$$

we have in view of (4) that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} X\left(f(mh)\right) = \kappa_X. \tag{5}$$

On the other hand,

$$\int_S X \, \mathrm{d}P_{N,w} = \frac{1}{U} \sum_{m=0}^{N} w(m) X\left(f(mh)\right). \tag{6}$$

Summing by parts, we find

$$\begin{aligned} \frac{1}{U} \sum_{m=0}^{N} w(m) X\left(f(mh)\right) \;=\; & \frac{w(N)}{U} \sum_{m=0}^{N} X\left(f(mh)\right) \\ & - \frac{1}{U} \int_0^N \sum_{0\leqslant m \leqslant u} X\left(f(mh)\right) \, \mathrm{d}w(u). \end{aligned} \tag{7}$$

It follows in virtue of (5) that

$$\sum_{0\leqslant m\leqslant u} X\left(f(mh)\right) = \sum_{0\leqslant m\leqslant [u]} X\left(f(mh)\right) = ([u]+1)\kappa_X + r(u)u,$$

where $r(u) \to 0$ as $u \to \infty$. Moreover, since $w(t)$ is non-increasing,

$$U = \sum_{m=0}^{N} w(m) \geqslant w(N)(N+1). \tag{8}$$

Therefore, this and (7) imply

$$\begin{aligned} \frac{1}{U} \sum_{m=0}^{N} w(m) X\left(f(mh)\right) &= \frac{w(N)}{U}\left((N+1)\kappa_X + r(N)N\right) \\ &\quad - \frac{1}{U} \int_0^N \left(([u]+1)\kappa_X + r(u)u\right) \, \mathrm{d}w(u) \\ &= \frac{w(N)}{U}\left((N+1)\kappa_X + (N+1)r(N+1)\right) \\ &\quad - \frac{w(u)}{U}([u]+1)\kappa_X \Big|_0^N + \frac{\kappa_X}{U} \int_0^N w(u) \, \mathrm{d}([u]+1) - \frac{1}{U} \int_0^N r(u)u \, \mathrm{d}w(u) \\ &= \mathrm{o}(1) + \frac{\kappa_X}{U} \sum_{m=0}^{N} w(m) - \frac{1}{U} \int_0^N r(u)u \, \mathrm{d}w(u) \end{aligned}$$

$$= \kappa_X + \mathrm{o}(1) - \frac{1}{U}\int_0^N r(u)u\,\mathrm{d}w(u) \tag{9}$$

as $N \to \infty$. Let $N_1 = N_1(N) \to \infty$ as $N \to \infty$ be chosen so that

$$\frac{1}{U}\int_0^{N_1} r(u)u\,\mathrm{d}w(u) = \mathrm{o}(1) \quad \text{and} \quad N_1 w(N_1) = \mathrm{o}(U)$$

as $N \to \infty$. Then we obtain that

$$\begin{aligned}
&\frac{1}{U}\int_0^N r(u)u\,\mathrm{d}w(u) = \frac{1}{U}\int_0^{N_1} r(u)u\,\mathrm{d}w(u) + \frac{1}{U}\int_{N_1}^N r(u)u\,\mathrm{d}w(u)\\
&= \mathrm{o}(1) + \frac{B}{U}\max_{u\in[N_1,N]}|r(u)|\int_{N_1}^N u\,\mathrm{d}w(u)\\
&= \mathrm{o}(1) + \frac{B}{U}\max_{u\in[N_1,N]}|r(u)|uw(u)\Big|_{N_1}^N + \frac{B}{U}\max_{u\in[N_1,N]}|r(u)|\int_0^N w(u)\,\mathrm{d}u.
\end{aligned} \tag{10}$$

Here B is a quantity bounded by a constant. It is easily seen that

$$\int_0^N w(u)\,\mathrm{d}u = BU.$$

This together with (8) and (10) shows that

$$\frac{1}{U}\int_0^N r(u)u\,\mathrm{d}w(u) = \mathrm{o}(1).$$

Therefore, (9) yields

$$\frac{1}{U}\sum_{m=0}^N w(m)X\left(f(mh)\right) = \kappa_X + \mathrm{o}(1)$$

as $N \to \infty$, and by (6) we have that

$$\lim_{N\to\infty}\int_S X\,\mathrm{d}P_{N,w} = \int_S X\,\mathrm{d}P$$

for every real bounded continuous function X on S. This means that the measure $P_{N,w}$ converges weakly to P as $N \to \infty$. The theorem is proved.

We will give one corollary of theorem. Suppose that the Matsumoto zeta-function $\varphi(s)$ is meromorphically continuable to the region D_2, all poles being included in a compact set. Moreover, we require that, for $\sigma > \varrho$, the estimates

$$\varphi(\sigma + it) = B|t|^\alpha, \quad |t| \geqslant t_0 > 0, \quad \alpha > 0,$$

and

$$\int_0^T |\varphi(\sigma+it)|^2\,\mathrm{d}t = BT, \quad T\to\infty,$$

should be satisfied. Let

$$\Omega=\prod_{m=1}^{\infty}\gamma_{p_m}, \quad \gamma_{p_m}=\{s\in\mathbb{C}\colon |s|=1\} \quad \text{for all } m.$$

On the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, where m_H denotes the Haar measure on $(\Omega,\mathcal{B}(\Omega))$, define an $H(D_2)$-valued random element $\varphi(s,\omega)$ by the formula

$$\varphi(s,\omega)=\prod_{m=1}^{\infty}\prod_{j=1}^{g(m)}\left(1-\frac{a_m^{(j)}\omega(p_m)}{p_m^{sf(j,m)}}\right)^{-1}, \quad s\in D_2,$$

where $\omega(p_m)$ is the projection of $\omega\in\Omega$ to coordinate space γ_{p_m}.

COROLLARY. *Suppose that the function $\varphi(s)$ satisfies all conditions stated above, and that $\exp\left\{\frac{2\pi k}{h}\right\}$ is irrational for all integers $k\neq 0$. If $w(t)$ is a continuous non-increasing function for $t\geqslant 0$ such that $\lim_{N\to\infty}U(N)=\infty$, then the probability measure*

$$\mu_{N,w}(\varphi(s+imh)\in A), \quad A\in\mathcal{B}(M(D_2)),$$

converges weakly to the distribution of the random element $\varphi(s,\omega)$ as $N\to\infty$.

Proof follows from the Theorem and Theorem of [3].

References

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Pastaba apie diskrečias ribines teoremas Matsumoto dzeta funkcijai

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Straisnyje įrodyta diskrečioji ribinė teorema su svoriu Matsumoto dzeta funkcijai.