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Zeta functions of cusp forms

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Let $F(z)$ be a holomorphic normalized eigenform of weight κ for the full modular group $SL(2,\mathbb{Z})$. In this case $F(z)$ has the Fourier series expansion

$$F(z)=\sum_{m=1}^{\infty} c(m)e^{2\pi imz}, \quad c(1)=1.$$

Denote by $s=\sigma+it$ a complex variable. Then the function

$$\varphi(s,F)=\sum_{m=1}^{\infty}\frac{c(m)}{m^s}$$

is called the zeta-function attached to the form $F(z)$. The latter series converges absolutely for $\sigma>\frac{\kappa+1}{2}$, and the function $\varphi(s,F)$ is holomorphic in this region. It is well known that the function $\varphi(s,F)$ is analytically continuable to an entire function. Moreover, $\varphi(s,F)$ has the Euler product expansion

$$\varphi(s,F)=\prod_{m=1}^{\infty}\left(1-\frac{\alpha(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)}{p^s}\right)^{-1}, \quad \sigma>\frac{\kappa+1}{2},$$

over all primes p with $\alpha(p)$ and $\beta(p)$ satisfying $c(p)=\alpha(p)+\beta(p)$. This and the Deligne estimates $|\alpha(p)|\leqslant p^{\frac{\kappa-1}{2}}$, $|\beta(p)|\leqslant p^{\frac{\kappa-1}{2}}$ show that $\varphi(s,F)\neq 0$ for $\sigma>\frac{\kappa+1}{2}$. Let $w\neq 0$ be an arbitrary complex number. Then we can define a branch of multi-valued function $\varphi^w(s,F)$, for $\sigma>\frac{\kappa+1}{2}$, by

$$\varphi^w(s,F)=\exp\{w\log\varphi(s,F)\}=\sum_{m=1}^{\infty}\frac{g_w(m)}{m^s},$$

where $g_w(m)$ is a multiplicative function,

$$g_w(p^k)=\sum_{l=0}^{k} d_w(p^l)\alpha^l(p)d_w(p^{k-l})\beta^{k-l}(p),$$

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and

$$d_w(p^k) = \frac{w(w+1)...(w+k-1)}{k!}, \quad k = 1, 2, ...$$

The investigation of the moments

$$\int_0^T |\varphi(\sigma + it)|^{2v} \,\mathrm{d}t, \quad \sigma \geqslant \frac{\kappa}{2}, \quad v \geqslant 0,$$

require to know the behaviour of the mean-value of $g_w(m)$. Let $h_w(m) = g_w^2(m)m^{1-\kappa}$. In this note we will consider the mean value $\sum_{m\leqslant x} h_w(m)$ as $x \to \infty$. We set $\varepsilon(x) = \max\left(|w|^{-2}\varepsilon_2(x), (\log x)^{-2}\right)$, where $\varepsilon_2(x) = \max(\varepsilon_1(x), (\log x)^{-1})$ and

$$\varepsilon_1(x) = \sup_{y\geqslant x} \left|y^{-1}\sum_{p\leqslant y} h_w(p)\log p - w^2\right| \to 0$$

as $x \to \infty$ by properties of $h_w(m)$. Denote

$$m(h_w, x) = \prod_{p\leqslant x} \big(1 + \sum_{\alpha=1}^{\infty} \frac{h_w(p^\alpha)}{p^\alpha}\big),$$

$$R(y) = \max\big(|w|^2, \frac{\log^2 y}{y}\big),$$

where $y \leqslant x$ and $\log^2 y \geqslant c\log x \log^2\log x$ with some $c > 0$. We suppose that w is a complex number such that $|w| \leqslant \frac{1}{2}$ and $Re w^2 > 0$. Let $t = \log x/\log y$, γ_0 stand for the Euler constant, and let B be a quantity bounded by a constant.

Theorem 1. *Suppose that* $\varepsilon(x) \to 0$ *as* $x \to 0$*. Then uniformly in* w, x *and* y

$$\begin{aligned}\sum_{m\leqslant x} h_w(m) \;&=\; \frac{\mathrm{e}^{-\gamma_0 w^2} x m(h_w, x) t^{o(|w|^2)}}{\Gamma(w^2)\log x}\left(1 + \frac{B|w|^2}{y}\right)\\ &+\; \frac{B|w|^2 x m(|h_w|, x) t^{-1+o(|w|^2)}}{\log x} + \mathrm{o}\left(\frac{x m(|h_w|, x) t^{|w|^2)} R(y)}{\log x}\right)\end{aligned}$$

as $x \to \infty$.

Note that in applications we usually need that $w = w(x) \to 0$ as $x \to \infty$. The uniformity in w allows us to consider this case.

We will give a sketch of the proof only, the details will be given elsewhere.

First we observe that $h_w(m)$ is a multiplicative function, $|h_w(m)| \leqslant 1$, moreover, $h_w(p) = w^2 c_p^2$, where $c_p = c(p)p^{\frac{1-\kappa}{2}}$. Applying one Rankin's result [1], hence we obtain that

$$\sum_{p \leqslant x} h_w(p) = \frac{w^2 x}{\log x}\bigl(1 + \mathrm{o}(1)\bigr), \quad x \to \infty. \tag{1}$$

Now we consider the mean values of $h_w(m)$ with large and small prime divisors. Let

$$S_1(x,y) = \sum_{m_1 \leqslant x} h_w(m_1),$$

where all prime divisors of m_1 are greater than y. Denote by $\rho(u), u \geqslant 0$, a continuous solution of the difference – differential equation $u\rho'(u) = w^2\rho(u-1)$ with initial condition $\rho(u) = 1$ for $0 \leqslant u \leqslant 1$. Then the application of known methods and (1) shows that uniformly in w, x and y

$$S_1(x,y) = 1 + \frac{x\rho'(t)}{\log y} - w^2 \frac{y}{\log y} + \frac{B|w|^2 x\varepsilon(y) t^{|w|^2-1} m(|h_w|, x)}{\log y m(|h_w|, y)} + \frac{Bx\varepsilon(y)\log y t^{|w|^2-1}}{y}. \tag{2}$$

Denote by m_2 positive integers free of prime divisors greater than y, and let

$$S_2(x,y) = \sum_{m_2 > x} \frac{h_w(m_2)}{m_2}.$$

Then (1) implies that uniformly in w, x and y

$$S_2(x,y) = Bm(|h_w|, y)\exp\{-t\log t + B|w|^2 t\}. \tag{3}$$

Now define

$$S_3(x,y) = \sum_{m_2 \leqslant x} h_w(m_2).$$

Then, using the estimates

$$\sum_{\substack{p^{\alpha+\beta} \leqslant x \\ \alpha,\beta \geqslant 1}} |h_w(p^\alpha)||h_w(p^\beta)| \log p^\alpha = B|w|^4\sqrt{x},$$

$$\sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 2}} |h_w(p^\alpha)| \log p^\alpha = B|w|^2\sqrt{x}, \quad x \to \infty,$$

we deduce that uniformly in w, x and y

$$S_3(x,y) = B|w|^2 xm(|h_w|,y)\exp\{-t\log t + B|w|^2 t\}. \tag{4}$$

Proof of the Theorem 1. First we observe that

$$\begin{aligned}\sum_{m\leqslant x} h_w(m) &= \sum_{m_2\leqslant x} h_w(m_2) \sum_{m_1\leqslant x/m_2} h_w(m_1)\\ &= \sum_{m_2\leqslant x/y} h_w(m_2)\left(S_1\left(\frac{x}{m_2},y\right)-1\right) + \sum_{m_2\leqslant x} h_w(m_2).\end{aligned} \tag{5}$$

By (2)we obtain that the first term in the right-hand side of (5) is

$$\begin{aligned}&\frac{x}{\log y}\sum_{m_2\leqslant x/y}\frac{h_w(m_2)}{m_2}\rho'\left(\frac{\log\frac{x}{m_2}}{\log y}\right) - \frac{w^2 y}{\log y}\sum_{m_2\leqslant x/y} h_w(m_2)\\ &+B\left(\frac{|w|^2 x\varepsilon(y)t^{|w|^2-1}m(|h_w|,x)}{\log y m(|h_w|,y)} + \frac{x\varepsilon(y)\log y t^{|w|^2-1}}{y}\right)\sum_{m_2\leqslant x/y}\frac{|h_w(m_2)|}{m_2}.\end{aligned} \tag{6}$$

Since $\rho'(1) = w^2$, it is not difficult to see that the first term in (6) can be written in the form

$$\frac{xm(h,y)}{\log y}\rho'(t) - \frac{w^2 x}{\log y}\sum_{m_2>x/y}\frac{h_w(m_2)}{m_2} - \frac{x}{\log y}\int_1^t\left(\sum_{m_2>x/y^u}\frac{h(m_2)}{m_2}\right)\rho''(u)\,\mathrm{d}u.$$

Therefore, by (5) and (6)

$$\begin{aligned}\sum_{m\leqslant x} h_w(m) &= \frac{xm(|h_w|,y)}{\log y}\rho'(t) - \frac{w^2 x}{\log y}\sum_{m_2>x/y}\frac{h_w(m_2)}{m_2}\\ &- \frac{x}{\log y}\int_1^t\left(\sum_{m_2>x/y^u}\frac{h_w(m_2)}{m_2}\right)\rho''(u)\,\mathrm{d}u - \frac{w^2 y}{\log y}\sum_{m_2\leqslant x/y} h_w(m_2)\\ &+ \sum_{m\leqslant x} h_w(m_2) + \frac{B|w|^2 x\varepsilon(y)t^{|w|^2-1}m(|h_w|,x)}{\log y}\\ &+ \frac{Bx\varepsilon(y)\log y t^{|w|^2-1}m(|h_w|,x)}{y}.\end{aligned} \tag{7}$$

From asymptotic properties of $\rho'(u)$ it follows that

$$\begin{aligned}\frac{xm(h_w,y)}{\log y}\rho'(t) &= \frac{\mathrm{e}^{-\gamma_0 w^2}xm(h_w,x)t^{\mathrm{o}(|w|^2)}}{\Gamma(w^2)\log x}\left(1+\frac{B|w|^2}{y}\right)\\ &+ \frac{B|w|^2 xm(|h_w|,x)t^{-1+\mathrm{o}(|w|^2)}}{\log x},\end{aligned} \tag{8}$$

where $\Gamma(w^2)$ denotes the gamma function. All other terms in (7) must be estimated. In view of (3) and (4) we find that

$$\frac{w^2 x}{\log y} \sum_{m_2 > x/y} \frac{h_w(m_2)}{m_2} = \frac{B|w|^2 x m(|h_w|, x)}{\log x} \exp\{-c_1 t\}, \tag{9}$$

$$\sum_{m_2 \leqslant x} h_w(m_2) = \frac{B|w|^2 x m(|h_w|, x)}{\log x} \exp\{-c_2 t\}, \tag{10}$$

$$\frac{w^2 x}{\log y} \sum_{m_2 \leqslant x/y} h_w(m_2) = \frac{B|w|^2 x m(|h_w|, x)}{\log x} \exp\{-c_3 t\} \tag{11}$$

with some positive constants c_1, c_2 and c_3. Also, properties of $\rho(u)$ give the estimate

$$\frac{x}{\log y} \int_1^t \Bigg(\sum_{m_2 > x/y^u} \frac{h_w(m_2)}{m_2} \Bigg) \rho''(u)\, \mathrm{d}u = \frac{B|w|^2 x m(|h_w|, x) t^{-1}}{\log x}. \tag{12}$$

Moreover,

$$\frac{|w|^2 x \varepsilon(y) t^{|w|^2 - 1} m(|h_w|, x)}{\log y} = \frac{|w|^2 x \varepsilon(y) t^{|w|^2} m(|h_w|, x)}{\log y},$$

$$\frac{x \varepsilon(y) \log y t^{|w|^2 - 1} m(|h_w|, x)}{y} = \frac{x \varepsilon(y) \log^2 y t^{|w|^2} m(|h_w|, x)}{y \log x}.$$

This and (8)–(12) together with (7) prove the theorem.

References

[1] R.A. Rankin, An Ω-result for the coefficients of cusp form, *Math.Annalen*, **203**, 239–250 (1973).

Parabolinių formų dzeta funkcijos

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Gauta parabolinių formų dzeta funkcijos laipsnių koeficientų vidurkių asimptotika.