Liet. matem. rink., T. 43, spec. nr., 682–686



# Asymptotic expansions in the large deviation zones for the distribution function of sums of random variables in the series scheme

Dovilė DELTUVIENĖ, Leonas SAULIS (VGTU)
*e-mail:* l.saulis@fm.vtu.lt

Let $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$, be a triangular array of independent random variables (r.v.) with means $\mathbf{E}\xi_j^{(n)} = 0$, and variances ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2 > 0$, $j = \overline{1, n}$.

Put

$$S_n = \sum_{j=1}^{n} \xi_j^{(n)}, \quad B_n^2 = \sum_{j=1}^{n} {\sigma_j^{(n)}}^2, \quad Z_n = B_n^{-1} S_n, \tag{1}$$

$$F_{Z_n}(x) = \mathbf{P}(Z_n < x), \quad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\,\mathrm{d}y, \tag{2}$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{1}{2}x^2 \Big\}, \quad \psi(x) = \varphi(x)\big(1 - \Phi(x)\big)^{-1}. \tag{3}$$

The target of our work is to obtain large deviation theorems and exponential inequalities for the function $\mathbf{P}(Z_n \geqslant x)$. First we have to get the estimate of the $k$th-order cumulant $\Gamma_k(Z_n)$ of the r.v. $Z_n$, defined by equality (1), where $\Gamma_k(X) := \frac{1}{i^k} \frac{d^k}{dt^k} \ln f_X(t)\big|_{t=0}$, $k = 1, 2, \ldots$. Here $f_X(t) = \mathbf{E}\exp\{itX\}$ is the characteristic function (ch.f.) of the r.v. $X$. In what follows $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $[m]$ is an integer part of number $m$, and $\theta_i$, $i = 1, 2, \ldots$ denote some quantity, not exceeding $1$ in absolute value.

Let us say that r.v's $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$, are subject to condition $(B_\gamma)$, if there exist quantities $\gamma \geqslant 0$ and $K_j^{(n)} > 0$ such that

$$\big|\mathbf{E}\big(\xi_j^{(n)}\big)^k\big| \leqslant (k!)^{1+\gamma}\big(K_j^{(n)}\big)^{k-2}{\sigma_j^{(n)}}^2, \quad k = 3, 4, \ldots. \tag{$B_\gamma$}$$

In order to obtain the asymptotic expansion of the distribution function $F_{Z_n}(x)$ of the r.v. $Z_n$ defined by equality (1), in large deviation zones, according to the General Lemma 1 [4], one must have the estimates of $k$th-order cumulants of the r.v. $Z_n$.

PROPOSITION 1. *If for r.v. $\xi_j^{(n)}$, $j = \overline{1, n}$, condition $(B_\gamma)$ is fulfilled, then for the $k$th-order cumulant $\Gamma_k(Z_n)$ of the r.v. $Z_n$, the estimate*

$$\big|\Gamma_k(Z_n)\big| \leqslant (k!)^{1+\gamma}\Delta_n^{2-k}, \quad k = 3, 4, \ldots, \tag{4}$$

$$\Delta_n = K_n^{-1} B_n, \quad K_n := 2 \max_{1\leqslant j\leqslant n} (K_j^{(n)} \vee \sigma_j^{(n)}), \tag{5}$$

*holds.*

*Proof* of the Proposition 1 is obtain in [2]. Next, suppose that the r.v. $\xi_j^{(n)}$, $j = \overline{1,n}$, densities $p_{\xi_j}(x)$ are bounded, i.e.,

$$\sup_x p_{\xi_j}(x) \leqslant C_j^{(n)} < \infty. \tag{$D$}$$

Let $\xi_j^{(n)}(h)$ be a r.v. conjugate to the $\xi_j^{(n)}$, $j = \overline{1,n}$, with the distribution density $p_{\xi_j(h)}(y) := \mathrm{e}^{hy} p_{\xi_j}(y) \Big( \int_{-\infty}^{\infty} \mathrm{e}^{hy} p_{\xi_j}(y)\,\mathrm{d}y \Big)^{-1}$. Denote $S_n(h) = \sum_{j=1}^n \xi_j(h)$, $Z_n(h) = B_n^{-1}(h)\big(S_n(h) - M_n(h)\big)$. Then, it is easy to check that

$$M_n(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \Gamma_k(S_n) h^{k-1}, \quad B_n^2(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \Gamma_k(S_n) h^{k-2}, \tag{6}$$

where the quantity $h = h(x)$ is the solution to the equation $x = M_n(h)/B_n$. Since the r.v's $\xi_j^{(n)}$, $j = \overline{1,n}$, are independent, we have

$$f_{Z_n(h)}(t) = \mathbf{E}\, \mathrm{e}^{itZ_n(h)} = \exp\left\{ -it \frac{M_n(h)}{B_n(h)} \right\} \prod_{j=1}^{n} f_{\xi_j(h)}\Big( \frac{t}{B_n(h)} \Big). \tag{7}$$

Let us denote $R_{n,\gamma} = \int_{T_{n,\gamma}}^{T_n} \big| f_{n,\gamma}^*(t) \big| \frac{\mathrm{d}t}{t}$, where

$$f_{n,\gamma}^*(t) = \begin{cases} \displaystyle\sum_{k=0}^{s} \left(\frac{3}{2}\right)^k \frac{x^k}{k!} f_{Z_n}^{(k)}(t), & \gamma > 0, \\ f_{Z_n(h)}(t), & \gamma = 0, \end{cases} \tag{8}$$

$s = 2[(1/2)(\Delta_n^2/18)^{1/(1+2\gamma)}] - 2$, $f_{Z_n}^{(0)}(t) = f_{Z_n}(t) = \mathbf{E}\exp\{itZ_n\}$ and

$$\begin{aligned} &T_{n,\gamma} := (3/8)(1 - x/\Delta_{n,\gamma})\Delta_{n,\gamma}, \quad 0 \leqslant x < \Delta_{n,\gamma}, \\ &\Delta_{n,\gamma} := c_\gamma \Delta_n^{1/(1+2\gamma)}, \quad c_\gamma = (1/6)(\sqrt{2}/6)^{1/(1+2\gamma)}, \end{aligned} \tag{9}$$

PROPOSITION 2. *If for the r.v.* $\xi_j^{(n)}$, $j = \overline{1,n}$, *with* $\mathbf{E}\xi_j^{(n)} = 0$ *and* ${\sigma_j^{(n)}}^2 > 0$, $j = \overline{1,n}$, *condition* $(B_\gamma)$ *is fulfilled, then for each integer* $l$, $l \geqslant 3$ *and* $T_n \geqslant T_{n,\gamma}$ *in the interval* $0 \leqslant x < \Delta_{n,\gamma}$, *the relation*

$$\begin{aligned} \frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} &= \exp\left\{ L_{n,m}^*(x) \right\} \left\{ \frac{\psi(x)}{\psi\big(u_n(x)\big)} \bigg( 1 + \sum_{\nu=1}^{l-3} P_{\nu,n}\big(u_n(x)\big) \bigg) + \theta_1(x+1) \right. \\ &\left. \times \bigg( \frac{c(l,\gamma,x)}{\Delta_n^{l-2}} + \frac{285\Delta_n \exp\big\{ -(1 - x/\Delta_{n,\gamma})\sqrt{\Delta_{n,\gamma}} \big\}}{(1 - x/\Delta_{n,\gamma})} + \frac{6q}{T_n} + R_{n,\gamma} \bigg) \right\} \end{aligned} \tag{10}$$

*holds. Here*

$$L^*_{n,m}(x) = \sum_{k=3}^{m} \lambda_{k,n} x^k, \quad m = \begin{cases} (1/\gamma) + l - 1, & \gamma > 0, \\ \infty\,, & \gamma = 0. \end{cases} \tag{11}$$

The coefficients $\lambda_{k,n}$ are found by formula $(6)$ $[4]$ and expressed by cumulants of r.v. $Z_n$. For this coefficients the estimate $|\lambda_{k,n}| \leqslant (2/k)(16/\Delta_n)^{k-2}\big((k+1)!\big)^{\gamma}$, $k = 3, 4, \ldots$ holds. The quantity $u_n(x) = x\Big(1 + \sum_{k=1}^{l-3} c_{k,n} x^k + \theta_2 c^*(l,\gamma)(x/\Delta_n)^{l-2}\Big)$, where $c^*(l,\gamma) = 736 l(l-1)\,(7/2)^{l-2}(l!)^{\gamma}$ and the coefficients $c_{k,n}$ are expressed by the cumulants of the r.v. $Z_n$ and found by formula $(11)$ $[4]$.

Polynomials $P_{\nu,n}\big(u_n(x)\big)$ are determined by relation $(104)$ $[4]$. In particular,

$$\begin{aligned} P_{1,n}\big(u_n(x)\big) &= -\frac{1}{2}\Gamma_3(Z_n)\frac{1}{x} + \frac{3}{2}\big(2\Gamma_4(Z_n) - 3\Gamma_3^2(Z_n)\big) + \frac{1}{48}\big(72\Gamma_5(Z_n) \\ &\quad -394\Gamma_3(Z_n)\Gamma_4(Z_n) + 267\Gamma_3^3(Z_n)\big)x + \ldots, \\ P_{2,n}\big(u_n(x)\big) &= \frac{1}{24}\big(3\Gamma_4(Z_n) - 5\Gamma_3^2(Z_n)\big) + \frac{1}{24}\big(3\Gamma_5(Z_n) \\ &\quad -16\Gamma_3(Z_n)\Gamma_4(Z_n) + 15\Gamma_3^3(Z_n)\big)x + \ldots. \end{aligned} \tag{12}$$

Full expressions of the quantities $c(l,\gamma,x)$ and $q$ are obtained by (9) and (58) [4], respectively.

Notice, that the proof of Proposition 2 can be easily derive on the basis of the proof of Proposition 2 [1], p. 294.

According to Proposition 2, we have to estimate the integral $R_{n,\gamma}$ defined by relation (8) for $\gamma = 0$ and $\gamma > 0$, respectively. The known estimates for characteristic functions and employment of condition (D) (V.Statulevičius [5]) will help us to achieve this aim. According to the definition of the quantity $\Delta_{n,\gamma}$ by relation (9) in the case $\gamma = 0$, we have

$$T_{n,0} = \big(1/8\big)\big(1 - x/\Delta_{n,0}\big)\Delta_{n,0}, \quad \Delta_{n,0} = c_0\Delta_n, \quad c_0 = (1/6)(\sqrt{2}/6). \tag{13}$$

**Theorem 1.** *If for the r.v.* $\xi_j^{(n)}$, *with* $\mathbf{E}\xi_j^{(n)} = 0$ *and* ${\sigma_j^{(n)}}^2 > 0$, $j = \overline{1,n}$, *conditions* $(B_\gamma)$ *with* $\gamma = 0$ *and* $(D)$ *are fulfilled, then in the Cramer zone* $0 \leqslant x < \Delta_{n,0}$, *the asymptotic expansion* (10) *holds. Moreover,*

$$\begin{aligned} R_{n,0} &\leqslant 684\,\mathrm{e}^4\pi\sqrt{2\pi}K_n \max_{1\leqslant r_i\leqslant n}\prod_{i=1}^{4} C_{r_i}^{1/4}\exp\Big\{-\frac{1}{24K_n^2}\sum_{k=1}^{n}\frac{1}{C_k^2}\Big\} \\ &\quad + \frac{\pi^2}{2T_{n,0}}\exp\Big\{-\frac{1}{\pi^2}T_{n,0}^2\Big\}. \end{aligned} \tag{14}$$

The *proof* of Theorem 1 is obtained in [6].

**Theorem 2.** *Let for the r.v.* $\xi_j^{(n)}$, *with* $\mathbf{E}\xi_j^{(n)} = 0$ *and* ${\sigma_j^{(n)}}^2 > 0$, $j = \overline{1,n}$, *conditions* $(B_\gamma)$ *with* $\gamma > 0$, $(D)$ *and*

$$\lim_{n\to\infty} \frac{1}{(1 \vee L_{1,n})\Delta_{n,\gamma}} \frac{1}{K_n^2} \sum_{j=1}^{n} \frac{1}{C_j^2} \geqslant d > 0 \tag{$S$}$$

*be fulfilled. Then in the Linnik power zones* $0 \leqslant x < \Delta_{n,\gamma}$, *the asymptotic expansion* (10) *holds. Moreover,*

$$\begin{aligned} R_{n,\gamma} \leqslant\ & c_4(\gamma)\big(K_n/\Delta_n\big) \max_{1\leqslant r_i\leqslant n} \prod_{i=1}^{4} C_{r_i}^{1/4} \bigg\{ -\frac{5d}{2K_n^2} \sum_{k=1}^{n} \frac{1}{C_k^2} \bigg\} \\ & + T_{n,\gamma}^2 \exp\bigg\{ -\frac{1}{16\pi^2} T_{n,\gamma}^2 \bigg\} + \frac{5\pi^2 x^2}{8} T_{n,\gamma} \exp\bigg\{ -\frac{1}{\pi^2} T_{n,\gamma}^2 \bigg\}, \end{aligned} \tag{15}$$

*where* $T_{n,\gamma}$, $K_n$ *are defined by equalities* (9), (5), *and* $c_4(\gamma) = 192\,\mathrm{e}^2\sqrt{2\pi} \cdot 24^\gamma$.

To prove Theorem 2, it is necessary, by Proposition 2, to estimate the integral $R_{n,\gamma}$, which is defined by equality (8), with $\gamma > 0$. Follow to the proof of the Theorem 2 [1], we get inequality (15).

**Theorem 3.** *For the r.v.* $Z_n$ *defined by equality* (1), *exponential inequalities*

$$\mathbf{P}\big\{ \pm Z_n \geqslant x \big\} = \begin{cases} \exp\Big\{ -(8\cdot 2^\gamma)^{-1} x^2 \Big\}, & 0 \leqslant x \leqslant 2\big(2^{\gamma^2}\Delta_n\big)^{\frac{1}{1+2\gamma}}, \\ \exp\Big\{ -\frac{1}{4}\big(x\Delta_n\big)^{\frac{1}{1+\gamma}} \Big\}, & x \geqslant 2\big(2^{\gamma^2}\Delta_n\big)^{\frac{1}{1+2\gamma}}, \end{cases} \tag{16}$$

If inequality (4) is valid, then $\big|\Gamma_k(Z_n)\big| \leqslant \big(k!/2\big)^{1+\gamma} 2^{1+\gamma} \Delta_n^{2-k}$. The proof follows from this inequality and Lemma 2.4 [3].

EXAMPLE. Let $\xi_j^{(n)} := a_{nj}\xi_j$, where $a_{nj}$ are nonnegative quantities, and $\xi_j$ are nonidentically distributed r.v's with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2$, $j = \overline{1,n}$. Now $S_n = \sum_{j=1}^n a_{nj}\xi_j$, $Z_n = B_n^{-1}S_n$, where $B_n^2 = \sum_{j=1}^n a_{nj}^2\sigma_j^2$. Suppose that there exist quantities $\gamma \geqslant$ and $K > 0$ such that $|\mathbf{E}\xi_j^k| \leqslant (k!)^{1+\gamma}K^{k-2}\sigma_j^2$, $k = 3, 4, \ldots$. Further, suppose that for r.v. $\xi_j$ there exists density $p_{\xi_j}(x) \leqslant C_j < \infty$. In this case, in condition $(B_\gamma)$ $K_j^{(n)} = a_{nj}K$, and $\sigma_j^{(n)} = a_{nj}\sigma_j$. Then the quantities defined by equalities (4) and $(D)$ are

$$K_n = 2\max_{1\leqslant j\leqslant n} a_{nj}\{K \vee \sigma_j\}, \quad C_j^{(n)} = a_{nj}^{-1}C_j. \tag{17}$$

## References

[1] L. Saulis, Asymptotic expansions of large deviations for sums of nonidentically distributed random variables, *Acta Appl. Math.*, **58**, 291–310 (1999).

[2] D. Deltuvienė, L. Saulis, Asymptotic expansion of the distribution density function for the sum of random variables in the series scheme in large deviation zones, *Acta Appl. Math.*, **78**, 87–97 (2003).
[3] L. Saulis, V. Statulevičius, *Limit Theorems for Large Deviations*, Kluwer Academic Publishers, Dordrecht, Boston, London (1991).
[4] L. Saulis, Asymptotic expansions in large deviation zones for the distribution function of random variables with cumulants of regular growth, *Lith. Math. J.*, **36**, 365–392 (1996).
[5] V. Statulevičius, Limit theorems for the density functions and asymptotic expansions for distributions of sums of independent random variables, *Theory Probab. Appl.*, **10**, 645–659 (1965).
[6] D. Deltuvienė, Asymptotic expansion for the distribution function of series scheme of random variables in the large deviation Cramer zone, *Lith. Math. J.*, **42**, 691–696 (2002).
[7] S.A. Book, A large deviation theorem, *Z. Wahrsch. Verw. Geb.*, **26** (1973).

## Atsitiktininių dydžių sumos serijų schemoje pasiskirstymo funkcijos asimptotiniai skleidiniai didžiųjų nuokrypių zonose

D. Deltuvienė, L. Saulis

Darbas skirtas nepriklausomų atsitiktinių dydžių (at.d.) $\xi_j^{(n)}$, $j = \overline{1, n}$ su vidurkiais $\mathbf{E}\xi_j^{(n)} = 0$ ir dispersijomis ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2 < \infty$ sumos pasiskirstymo funkcijos asimptotinių skleidinių gavimui didžiųjų nuokrypių Kramerio ir laipsninėse Liniko zonose. Rezultatas gautas remiantis L. Saulio bendrąja Lema 1 [4], apjungiant charakteristinių funkcijų ir kumuliantų metodus. Darbas atskiru atveju praplečia at.d. sumavimo rezultatus gautus S.A. Book [7].