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The discounted limit theorems for large deviations

Leonas SAULIS (VGTU, MII), Dovilė DELTUVIENĖ (VGTU)
e-mail: lsaulis@fm.vtu.lt

1. Introduction and summary

Let $X_0, X_1, X_2, \ldots$ be a sequence of independent random variables (r.v.) with common distribution function $F(x)$. Let v be a discount factor $(0 < v < 1)$. Then we define

$$S_v = \sum_{k=0}^{\infty} v^k X_k, \tag{1.1}$$

wich may by interpreted as the present value of a sum of certain periodic and identically distributed payments X_k. We assume that the first three moments of X_k are finite:

$$\begin{aligned} &\mu = \int_{-\infty}^{\infty} x \,\mathrm{d}F(x) < \infty, \quad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \,\mathrm{d}F(x) < \infty, \\ &\rho = \int_{-\infty}^{\infty} |x-\mu|^3 \,\mathrm{d}F(x) < \infty. \end{aligned} \tag{1.2}$$

Then it is easy to see that the mean and variance of the r.v. S_v are

$$\mathbf{E}S_v = \mu(1-v)^{-1}, \quad \mathbf{D}S_v = \sigma^2(1-v^2)^{-1}, \tag{1.3}$$

respectively. It has been shown that the normalized random variable

$$Z_v = \sigma^{-1}(1-v)^{\frac{1}{2}}\big(S_v - \mu(1-v)^{-1}\big) \tag{1.4}$$

with the mean $\mathbf{E}Z_v = 0$ and variance $\mathbf{D}Z_v = (1+v)^{-1}$ is asymptotically normal for $v \to 1$. We denote the distribution function of the r.v. Z_v as $F_v(x)$ and that of the normal distribution with zero mean and variance $(1+v)^{-1}$ by

$$N_v(x) = \left(\frac{1+v}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{x} \exp\Big\{-\frac{1+v}{2}\Big\} y^2 \,\mathrm{d}y. \tag{1.5}$$

Hans U. Gerber in [1] has proved the discounted version of the Berry–Esseen theorem: if (1.2) holds, then for all x,

$$\big|F_v(x) - N_v(x)\big| \leqslant 5.4(\rho/\sigma^3)(1-v)^{\frac{1}{2}}. \tag{1.6}$$

We consider the asymptotic behaviour of probability $\mathbf{P}(Z_v \geqslant x)$ as $x = x_v \to \infty$, $v \to 1$, i.e., we prove large deviation theorems for the r.v. Z_v defined by (1.4) employing the cumulant method when centered moments $\mathbf{E}(X_0 - \mu)^s$ of the r.v. X_0 satisfy the generalized N.S. Bernstein condition: there exist quantities $\gamma \geqslant 0$ and $K > 0$ such that

$$\left|\mathbf{E}(X_0 - \mu)^s\right| \leqslant (s!)^{1+\gamma} K^{s-2}\sigma^2, \quad s = 3, 4, \ldots . \tag{B_γ}$$

2. The discounted version of large deviations

Denote

$$\Delta_v = \frac{\sigma}{2(K \vee \sigma)}(1-v)^{-\frac{1}{2}}, \quad \Delta_v(\gamma) = c_v(\gamma)\Delta_v^{\frac{1}{1+2\gamma}}, \tag{2.1}$$

where $c_v(\gamma) = (1/6)\big(\sqrt{2}/\big(6(1+v)^{1+\gamma}\big)\big)^{1/(1+2\gamma)}$ and $a \vee b = \max\{a, b\}$. In what follows, let θ with or without index denote some quantity, not always the same, not exceeding 1 in absolute value. Let $[m]$ be the integer part of the number m.

Theorem 1. *Let identically distributed r.v. X_k with $\mathbf{E}X_k = \mu$ and $\sigma^2 = \mathbf{E}(X-\mu)^2$, $k = 0, 1, 2, \ldots$ satisfy condition (B_γ). Then, for the distribution function $F_v(x)$ of the r.v. Z_v defined by* (1.4) *the relations of large deviations*

$$\begin{aligned} \frac{1 - F_v(x)}{1 - N_v(x)} &= \exp\left\{L_\gamma(x)\right\}\Big(1 + \theta_1 f(x)\frac{x+1}{\Delta_v(\gamma)}\Big), \\ \frac{F_v(-x)}{N_v(-x)} &= \exp\left\{L_\gamma(-x)\right\}\Big(1 + \theta_2 f(x)\frac{x+1}{\Delta_v(\gamma)}\Big) \end{aligned} \tag{2.2}$$

are valid in the interval $0 \leqslant x < \Delta_v(\gamma)$. Here

$$\begin{aligned} f(x) &= \frac{60\big(1 + (1+v)\Delta_v^2(\gamma)\exp\big\{-\big(1 - x/\Delta_v(\gamma)\big)\sqrt{\Delta_v(\gamma)}\big\}\big)}{(1 - x/\Delta_v(\gamma))}, \\ L_\gamma(x) &= \sum_{3 \leqslant k < p} \lambda_k x^k + \theta\big(x/\Delta_v(\gamma)\big), \quad p = \begin{cases} (1/\gamma) + 2, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases} \end{aligned} \tag{2.3}$$

The coefficients λ_k are expressed in terms of cumulants of the r.v. Z_v given by the formula $\lambda_k = -b_{k-1}/k$, where b_k are determined recursively from the equations

$$\sum_{r=1}^{j} \frac{1}{r!}\Gamma_{r+1}(Z_v) \sum_{j_1+\ldots+j_r=j} \prod_{i=1}^{r} b_{j_i} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \ldots . \end{cases} \tag{2.4}$$

In particular,

$$b_1 = \Gamma_2^{-1}(Z_v) = 1 + v, \quad b_2 = -\frac{1}{2}(1+v)^3\Gamma_3(Z_v),$$

$$b_3 = -\frac{1}{6}(1+v)^4\big(\Gamma_4(Z_v) - 3(1+v)\Gamma_3^2(Z_v)\big),$$
$$b_4 = -\frac{1}{24}(1+v)^5\big(\Gamma_5(Z_v) - 10(1+v)\Gamma_3(Z_v)\Gamma_4(Z_v) + 15(1+v)^2\Gamma_3^3(Z_v)\big),$$

where $\Gamma_k(Z_v)$, $k = 2, 3, \ldots$, *are defined by formula* (3.3). *For the coefficients* λ_k, *the estimate*

$$\big|\lambda_k\big| \leqslant \frac{2(1+v)}{k}\left(\frac{16(1+v)}{\Delta_v}\right)^{k-2}\big((k+1)!\big)^{\gamma}, \quad k = 3, 4, \ldots \tag{2.5}$$

holds, and therefore

$$L_\gamma(x) \leqslant \frac{(1+v)x^2}{2}\frac{x}{x+8\Delta_v(\gamma)}, \quad L_\gamma(-x) \geqslant -\frac{(1+v)x^3}{3\Delta_v(\gamma)}. \tag{2.6}$$

Theorem 2. *Let a r.v.* X_k, $k = 0, 1, 2, \ldots$ *satisfy condition* (B_γ). *Then, for distribution function* $F_v(x)$, *the relations*

$$\lim_{v\to 1}\frac{1-F_v(x)}{1-N_v(x)} = 1, \quad \lim_{v\to 1}\frac{F_v(-x)}{N_v(-x)} = 1 \tag{2.7}$$

hold for $x \geqslant 0$, $x = o\big(\Delta_v^{\nu}\big)$, *where* Δ_v *is defined by formula* (2.1) *and* $\nu = \nu(\gamma) = (1 + 2\max\{1,\gamma\})^{-1}$.

In particular, if $\gamma = 0$, *then relations* (2.7) *hold for* $x \geqslant 0$, $x = \mathrm{o}\big((1-v)^{-\frac{1}{6}}\big)$.

Theorem 3. *Let a r.v.* X_k, $k = 0, 1, 2, \ldots$ *satisfy condition* (B_γ). *Then for probability* $\mathbf{P}\big(\pm Z_v \geqslant x\big)$ *of the r.v.* Z_v *defined by equality* (1.4), *the estimates*

$$\mathbf{P}\big(\pm Z_v \geqslant x\big) \leqslant \begin{cases} \exp\big\{-\frac{1}{4H_v}x^2\big\}, & 0 \leqslant x \leqslant \big(H_v^{1+\gamma}\Delta_v\big)^{1/(1+2\gamma)}, \\ \exp\big\{-\frac{1}{4}\big(x\Delta_v\big)^{1/(1+\gamma)}\big\}, & x \geqslant \big(H_v^{1+\gamma}\Delta_v\big)^{1/(1+2\gamma)}, \end{cases} \tag{2.8}$$

are valid with $H_v = 2^{1+\gamma}(1+v+v^2)^{-1}$ *and* $\Delta_v = \frac{\sigma}{2(K\vee\sigma)}(1-v)^{-\frac{1}{2}}$.

3. Proof of Theorems 1–3

The theorems are proved by the cumulant method which was proposed by V. Statulevičius and extended by R. Rudzkis, L. Saulis, V. Statulevičius in [3]–[5]. Denote the characteristic function of the r.v. X_k, by $f(t) = \mathbf{E}\exp\{itX_k\}$. Then, recalling that the r.v. S_v was defined by equality (1.1) and taking into consideration that the r.v. X_k, $k = 0, 1, 2, \ldots$ are independent, we obtain the expression of the characteristic function $f_{Z_v}(t)$ of Z_v

$$f_{Z_v}(t) = \mathbf{E}\exp\{itZ_v\} = \mathbf{E}\exp\big\{it\sigma^{-1}(1-v)^{\frac{1}{2}}\big(S_v - \mu(1-v)^{-1}\big)\big\}$$

$$= \exp\left\{-it\mu\sigma^{-1}(1-v)^{-\frac{1}{2}}\right\}\mathbf{E}\exp\left\{it\sigma^{-1}(1-v)^{\frac{1}{2}}S_v\right\}$$
$$= \exp\left\{-it\mu\sigma^{-1}(1-v)^{-\frac{1}{2}}\right\}\prod_{k=0}^{\infty} f\left(\sigma^{-1}(1-v)^{\frac{1}{2}}v^k t\right). \tag{3.1}$$

Then

$$\ln f_{Z_v}(t) = -it\mu\sigma^{-1}(1-v)^{-1/2} + \sum_{k=0}^{\infty}\ln f\left(\sigma^{-1}(1-v)^{\frac{1}{2}}v^k t\right). \tag{3.2}$$

The s-th order cumulant of the r.v. Z_v is defined by

$$\Gamma_s(Z_v) := \frac{1}{i^s}\frac{d^s}{dt^s}\ln f_{Z_v}(t)\Big|_{t=0}, \quad s = 1, 2, \ldots. \tag{3.3}$$

Next, employing (3.2), we obtain $\Gamma_1(Z_v) = \mathbf{E}Z_v = 0$,

$$\Gamma_s(Z_v) = \left(\frac{(1-v)^{1/2}}{\sigma}\right)^s \frac{1}{1-v^s}\,\Gamma_s(X_0), \quad s = 2, 3, \ldots. \tag{3.4}$$

In particular, $\Gamma_2(Z_v) = \mathbf{D}Z_v = (1+v)^{-1}$ and

$$\Gamma_3(Z_v) = \frac{(1-v)^{1/2}}{1+v+v^2}\,\frac{\Gamma_3(X_0)}{\sigma^3} = \frac{(1-v)^{1/2}}{1+v+v^2}\,\frac{\mathbf{E}(X_0-\mu)^3}{\sigma^3}, \ldots. \tag{3.5}$$

Our next step is to obtain the majorating upper estimates for the s-th order cumulants $\Gamma_s(Z_v),\ s = 3, 4, \ldots$ of the r.v. Z_v.

PROPOSITION 3.1. If for a r.v. X_k, $k = 0, 1, 2, \ldots$ with $\mathbf{E}X_k = \mu$ and $\sigma^2 = \mathbf{E}(X_k-\mu)^2$ the condition (B_γ) is fulfilled, then

$$\left|\Gamma_s(Z_v)\right| \leqslant \frac{1}{1+v+v^2}\,\frac{(s!)^{1+\gamma}}{\Delta_v^{s-2}}, \quad s = 3, 4, \ldots \tag{3.6}$$

where Δ_v is defined by (2.1).

Proof. Making use of Lemma 3.1 in [3] or [4] and noting that $\Gamma_s(X_0 - \mathbf{E}X_0) = \Gamma_s(X_0),\ s = 2, 3, \ldots$, from condition (B_γ) we obtain

$$\left|\Gamma_s(X_0)\right| \leqslant (s!)^{1+\gamma}\left(2(K\vee\sigma)\right)^{s-2}\sigma^2, \quad s = 3, 4, \ldots. \tag{3.7}$$

Now, applying expression (3.4) of cumulants $\Gamma_s(Z_v),\ s = 3, 4, \ldots$, we obtain

$$\left|\,\Gamma_s(Z_v)\right| \leqslant \frac{(s!)^{1+\gamma}}{1-v^s}\left(\frac{(1-v)^{1/2}}{\sigma}\right)^s\left(2(K\vee\sigma)\right)^{s-2}\sigma^2$$
$$\leqslant \frac{(1-v)^{s/2}(s!)^{1+\gamma}}{1-v^3}\left(\frac{2(K\vee\sigma)}{\sigma}\right)^{s-2} = \frac{1}{1+v+v^2}\,\frac{(s!)^{1+\gamma}}{\Delta_v^{s-2}}, \tag{3.8}$$

where Δ_v is defined by (2.1).

Proof of Theorem 1. First note that $F_v(x) = \mathbf{P}(Z_v < x) = \mathbf{P}(Z_v^* < x_v)$, where $Z_v^* = (\mathbf{D}Z_v)^{-1/2} Z_v = (1+v)^{1/2} Z_v$ and $x_v = (1+v)^{1/2} x$. Having in mind that $\mathbf{E}Z_v^* = 0\,, \mathbf{D}Z_v^* = 1$ and $\Gamma_s(Z_v^*) = (1+v)^{s/2}\Gamma_s(Z_v)$, $s = 1, 2, \ldots$ and applying estimate (3.6), we obtain

$$\bigl|\Gamma_s(Z_v^*)\bigr| \leqslant \frac{(1+v)^{s/2}}{1+v+v^2}\,\frac{(s!)^{1+\gamma}}{\Delta_v^{s-2}} \leqslant \frac{(s!)^{1+\gamma}}{(\Delta_v^*)^{s-2}}, \quad s = 3, 4, \ldots, \tag{3.9}$$

where $\Delta_v^* = (1+v)^{-1/2}\Delta_v$. It is obvious that

$$\Phi(x_v) = (2\pi)^{-1/2}\int_{-\infty}^{x_v} \exp\Big\{-\frac{1}{2}\,y^2\Big\}\,\mathrm{d}y = N_v(x), \tag{3.10}$$

where $N_v(x)$ is defined by (1.5). Hence, the r.v. $\xi = Z_v^*$ satisfies the condition (S_γ) in Lemma (2.3) (see [3], [4]) as $\Delta = \Delta_v^* = (1+v)^{-1/2}\Delta_v$. Consequently, basing on this lemma and relations (3.7), (3.8), we arrive at the statement of Theorem 1.

Proof of Theorem 2. At first note that

$$\Delta_v = \frac{\sigma}{2(K\vee\sigma)}\frac{1}{(1-v)^{1/2}} \longrightarrow \infty, \quad v \to 1. \tag{3.11}$$

Now, relying on the statement of Theorem 1, we can get convinced that for all $x = o\bigl(\Delta_v^\nu\bigr)$, where $\nu = \nu(\gamma) = (1 + 2\max\{1,\gamma\})^{-1}$,

$$\Delta_v^{-1}(\gamma)\,x = c_v^{-1}(\gamma)\; o\Bigl(\Delta_v^{2(\gamma-\max\{1,\gamma\})/\bigl((1+2\gamma)(1+2\max\{1,\gamma\})\bigr)}\Bigr) \to 0, \tag{3.12}$$

because $\gamma - \max\{1,\gamma\} \leqslant 0$ for all $\gamma \geqslant 0$. So, according to Theorem 1 we need to prove that $L_\gamma(x) \to 0$ for all $x = o\bigl(\Delta_v^\nu\bigr)$. Recalling expression (2.3) of $L_\gamma(x)$ and making use of estimates (3.5) of the cumulants $\Gamma_s(Z_v)$, we derive

$$\begin{aligned}\bigl|\lambda_3\,x^3\bigr| &= \frac{1}{6}(1+v)^3\,\bigl|\Gamma_3(Z_v)x^3\bigr| \leqslant \frac{1+v)^2\,6^\gamma}{\Delta_v} o(\Delta_v^{3\nu}) \\ &= (1+v)^2\,6^\gamma\; o\bigl(\Delta_v^{2(1-\max\{1,\gamma\})/(1+2\max\{1,\gamma\})}\bigr) \to 0\,,\end{aligned}$$

because $1 - \max\{1,\gamma\} \leqslant 0$.

Proof of Theorem 3. First, note that $\Gamma_2(Z_v) = \mathbf{D}Z_v = (1+v)^{-1} < H_v$, where $H_v = 2^{1+\gamma}(1+v+v^2)^{-1}$. Then, using estimates (3.5) of the s-th order cumulants $\Gamma_s(Z_v)$, $s \geqslant 3$, of the r.v. Z_v with the mean $\mathbf{E}Z_v = 0$, we obtain

$$\bigl|\Gamma_s(Z_v)\bigr| \leqslant \Big(\frac{s!}{2}\Big)^{1+\gamma}\frac{H_v}{\Delta_v^{s-2}}, \quad s = 2, 3, \ldots. \tag{3.13}$$

Thus the r.v. $\xi = Z_v$ satisfies condition (2.12) of Lemma 2.4 in [3] or [4] as $H = H_v$ and $\bar\Delta = \Delta_v$, where Δ_v is defined by (2.1).

References

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Diskontavimo ribinės teoremos atsižvelgiant į didžiuosius nuokrypius

L. Saulis, D. Deltuvienė

Darbe gautos atitinkamai normuotos sumos $S_v = \sum_{k=0}^{\infty} v^k X_k,\ 0 \leqslant v < 1$ skirstinio didžiųjų nuokrypių teoremos ir eksponentinės nelygybės, kai atsitiktiniai dydžiai $X_0, X_1, X_2, \ldots$ tenkina N.S. Bernšteino sąlygą.