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On the denseness in the space of analytic functions

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For any region G on the complex plane, by $H(G)$ we denote the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $G_1, \ldots, G_n$ be simply connected regions on $\mathbb{C}$, and $H(G_1, \ldots, G_n) = H(G_1) \times \ldots \times H(G_n)$. In the theory of zeta-functions we have often to consider the denseness in $H(G_1, \ldots, G_n)$ of some series. For this aim the following statement is useful. Denote by $\mathcal{B}(\mathbb{C})$ the class of Borel sets of $\mathbb{C}$.

Theorem. *Let* $\{\underline{f}_m\} = \{(f_{1m}, \ldots, f_{nm})\}$ *be a sequence in* $H(G_1, \ldots, G_n)$ *which satisfies:*

1^0 *If* $\mu_1, \ldots, \mu_n$ *are complex measures on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *with compact supports contained in* $G_1, \ldots, G_n$, *respectively, such that*

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} f_{jm} \, \mathrm{d}\mu_j \right| < \infty,$$

then

$$\int_{\mathbb{C}} s^r \, \mathrm{d}\mu_j(s) = 0$$

for $j = 1, \ldots, n$, $r = 0, 1, 2, \ldots$;

2^0 *The series*

$$\sum_{m=1}^{\infty} \underline{f}_m$$

converges in $H(G_1, \ldots, G_n)$;

3^0 *For any compacts* $K_1 \subseteq G_1, \ldots, K_n \subseteq G_n$,

$$\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K_j} \left| f_{jm}(s) \right|^2 < \infty.$$

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m \underline{f}_m$$

with $|a_m| = 1$, $m = 1, 2, \ldots$, *is dense in* $H(G_1, \ldots, G_n)$.

Note that the case $n = 1$ was considered in [2], Theorem 6.3.10, and for $G_1 = \ldots = G_n$ the theorem was obtained by B. Bagchi [1]. However, for example, the investigation of the joint universality of zeta-functions attached to cusp forms of different weight require to consider the case of the space $H(G_1, \ldots, G_n)$.

Proof of the theorem. Let K_j be a compact subset of G_j, $j = 1, \ldots, n$. We choose a simply connected region V_j such that $K_j \subseteq V_j$, the closure $\overline{V}_j$ is a compact subset of G_j and the boundary ∂V_j of V_j, $j = 1, \ldots, n$, is an analytic simple closed curve. Consider the Hardy space $H^2(V_j)$, see the definition in [2], Section 6.6.3, which is an Hilbert space, $j = 1, \ldots, n$. Now let

$$H^2(V_1, \ldots, V_n) = H^2(V_1) \times \ldots \times H^2(V_n).$$

Define for $\underline{f} = f(f_1, \ldots, f_n)$, $\underline{g} = (g_1, \ldots, g_n) \in H^2(V_1, \ldots, V_n)$ the inner product by

$$(\underline{f}, \underline{g}) = \sum_{j=1}^{n} (f_j, g_j),$$

where (f_j, g_j) is the inner product on $H^2(V_j)$, $j = 1, \ldots, n$. Thus we have that $H^2(V_1, \ldots, V_n)$ is an Hilbert space again. By the proof of Theorem 6.3.10 from [2] we have

$$\|\underline{f}_m\|^2 = (\underline{f}_m, \underline{f}_m) = \sum_{j=1}^{n} (f_j, f_j) = \sum_{j=1}^{n} \|f_j\|^2 = B \sum_{j=1}^{n} \sup_{s \in V_j} |f_j(s)|^2.$$

Here B is a quantity bounded by a constant. Therefore in view of the condition 3^0, by the choice of V_j,

$$\sum_{m=1}^{\infty} \|\underline{f}_m\|^2 < \infty.$$

Now let $\underline{g} \in H^2(V_1, \ldots, V_n)$ be such that

$$\sum_{m=1}^{\infty} |(\underline{f}_m, \underline{g})| < \infty. \tag{1}$$

Using the formula for the inner product in $H^2(V_j)$, see [2], Section 6.6.3, we obtain that there exists a complex measure μ_j on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with support contained in ∂V_j, $j = 1, \ldots, n$, such that

$$(\underline{f}_m, \underline{g}) = \sum_{j=1}^{n} \int_{\mathbb{C}} f_{jm} \, d\mu_j.$$

Hence by (1)

$$\sum_{m=1}^{\infty}\left|\sum_{j=1}^{n}\int_{\mathbb{C}} f_{jm}\,\mathrm{d}\mu_j\right| < \infty,$$

and therefore by the hypothesis 1^0 of the theorem

$$\int_{\mathbb{C}} s^r\,\mathrm{d}\mu_j = 0$$

for $j = 1, \ldots, n$, $r = 0, 1, 2, \ldots$. This means that g_j is orthogonal to all the polynomials, $j = 1, \ldots, n$. Since the polynomials are dense in the topology of $H^2(V_j)$, hence we obtain that $g_j = 0$, $j = 1, \ldots, n$, and $\underline{g} = \underline{0}$. Consequently,

$$\sum_{m=1}^{\infty} |(\underline{f}_m, \underline{g})| = \infty$$

for $\underline{0} \neq \underline{g} \in H^2(V_1, \ldots, V_n)$. Therefore by Theorem 6.1.16 from [2] we obtain that the set of all convergent series in $H^2(V_1, \ldots, V_n)$

$$\sum_{m=1}^{\infty} \alpha_m \underline{f}_m$$

with $|\alpha_m| = 1$, $m = 1, 2, \ldots$, is dense in $H^2(V_1, \ldots, V_n)$. Let $\underline{f} = (f_1, \ldots, f_n) \in H(G_1, \ldots, G_n)$ and $\varepsilon > 0$. Since the convergence in the $H^2(V_j)$ topology implies the uniform convergence on compact subsets of V_j, $j = 1, \ldots, n$, hence we deduce that there exists a sequence $\{\alpha_m, |\alpha_m| = 1\}$ such that the series

$$\sum_{m=1}^{\infty} \alpha_m f_{jm}$$

converges uniformly on K_j for all $j = 1, \ldots, n$, and

$$\sum_{j=1}^{n} \sup_{s\in K_j}\left|\sum_{m=1}^{\infty} \alpha_m f_{jm}(s) - f_j(s)\right| < \frac{\varepsilon}{4}.$$

Hence there exists a natural number M such that

$$\sum_{j=1}^{n} \sup_{s\in K_j}\left|\sum_{m=1}^{M} \alpha_m f_{jm}(s) - f_j(s)\right| < \frac{\varepsilon}{2}, \tag{2}$$

and, in view of the hypothesis 2^0 of the theorem,

$$\sum_{j=1}^{n} \sup_{s\in K_j}\left|\sum_{m=M+1}^{\infty} f_{jm}(s)\right| < \frac{\varepsilon}{2}. \tag{3}$$

Let

$$a_m = \begin{cases} \alpha_m, & 1 \leqslant m \leqslant M, \\ 1, & m > M. \end{cases}$$

Then (2) and (3) yield

$$\sum_{j=1}^{n} \sup_{s \in K_j} \left| \sum_{m=1}^{\infty} a_m f_{jm}(s) - f_j(s) \right| < \varepsilon,$$

and the theorem is proved.

Let $F_j(z)$ be a holomorphic cusp form of weight κ_j for the full modular group $SL(2, \mathbb{Z})$, and we assume that $F_j(z)$ is a normalized eigenform, $j = 1, \ldots, n, n \geqslant 2$. Consider the zeta-functions

$$\varphi(s, F_j) = \sum_{m=1}^{\infty} c_j(m) m^{-s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}, \quad \Re s > \frac{\kappa_j + 1}{2},$$

$j = 1, \ldots, n$, and their analytic continuation. Here $c_j(m)$ denote the coefficients of the Fourier series expansion for $F_j(z)$, and $c_j(p) = \alpha_j(p) + \beta_j(p), j = 1, \ldots, n$.

Let, for $|z| = 1$,

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots,$$

and define

$$\underline{f}_p(s_1, \ldots, s_n, a_p) = \left(-\log\left(1 - \frac{\alpha_1(p) a_p}{p^{s_1}}\right) - \log\left(\frac{\beta_1(p) a_p}{p^{s_1}}\right), \ldots, \log\left(\frac{\alpha_n(p) a_p}{p^{s_n}}\right) - \log\left(\frac{\beta_n(p) a_p}{p^{s_n}}\right)\right),$$

where $|a_p| = 1$ for all primes p, and for some $N > 0$,

$$s_j \in D_{j,N} = \left\{ s \in \mathbb{C} : \frac{\kappa_j}{2} < \Re s < \frac{\kappa_j + 1}{2}, \ |t| < N \right\}, \qquad j = 1, \ldots, n.$$

Then the theorem can be applied to prove that the set of all convergent series

$$\sum_p \underline{f}_p(s_1, \ldots, s_n, a_p)$$

is dense in $H_n = H(D_{1,N}) \times \ldots \times H(D_{n,N})$.

References

[1] B. Bagchi, *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and other Allied Dirichlet Series*, Ph. D. Thesis, Calcuta, Indian Statistical Institute (1981).

[2] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht/Boston/London (1996).

Apie tirštumą analizinių funkcijų erdvėje

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Nurodyta pakankama sąlyga, kad konverguojančių eilučių aibė analizinių funkcijų erdvėje būtų tiršta toje erdvėje.