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On uniform error of kernel estimate of discontinuous regression function

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1. Introduction and results

Suppose the measurements Y_i at fixed design points $t_i = i/n$, $i = 1, \dots, n$, satisfy the following regression model:

$$Y_i = g(t_i) + \epsilon_i, \quad 1 \leqslant i \leqslant n, \tag{1}$$

where g is an unknown regression function and ϵ_i are independent identically distributed errors with zero mean $E(\epsilon_i) = 0$ and finite variance $\operatorname{var}(\epsilon_i) = \sigma^2 < \infty$. We will assume that the function g has a jump of magnitude Δ at a change-point τ, $0 < \tau < 1$ and is smooth elsewhere. More precisely we assume that

$$g(t) = f(t) + \Delta 1_{[\tau,1]}(t), \quad t \in [0,1],$$

where f is a continuous function. We will consider the case where neither τ nor Δ are known. For this model Kee–Hoon Kang *et all.* [1] suggested the following procedure of kernel estimation of the function g. Using preliminary estimators of τ and Δ, say $\widehat{\tau}$ and $\widehat{\Delta}$ respectively, one computes the adjusted data

$$Y_i^* = Y_i - \widehat{\Delta} 1_{[\hat{\tau},1]}(t_i), \quad 1 \leqslant i \leqslant n.$$

The regression function g is estimated then by

$$\widehat{g}^*(t) = \frac{1}{h} \sum_{i=1}^{n} Y_i^* \int_{s_{i-1}}^{s_i} K^*\Big(\frac{t-u}{h}\Big) \mathrm{d}u + \widehat{\Delta} 1_{[\hat{\tau},1]}(t), \tag{2}$$

where $s_0 = 0$, $s_n = 1$, $s_k = (t_k + t_{k+1})/2$, $k = 1, \dots, n-1$. The kernel K^* is defined by (denoting $v = (t-u)/h$):

$$K^*(v) = \begin{cases} K_+(v,q), & \text{for } 0 \leqslant t \leqslant h \text{ with } q = \dfrac{t}{h}, \\ K(v), & \text{for } h \leqslant t \leqslant 1-h, \\ K_-(v,q), & \text{for } 1-h \leqslant t \leqslant 1 \text{ with } q = \dfrac{1-t}{h}, \end{cases}$$

where, $K_+(\cdot, q)$ and $K_-(\cdot, q)$ are kernel functions of order d with supports $[-1, q]$ and $[-q, 1]$ respectively, K is a kernel function of order d with support $[-1, 1]$ and $h = h(n)$ is a sequence of bandwidths which is required to satisfy $h \to 0, nh \to \infty$ as $n \to \infty$.

Particularly, in [1] it is proved that under certain smoothness conditions on the kernels K, $K_\pm(\cdot, q)$ the integrated square error of the estimator $\widehat{g}^*$ computed with Müller's [2] estimators $\widehat{\tau}$ and $\widehat{\Delta}$ of τ and Δ respectivly is

$$\int_0^1 \left|\widehat{g}^*(t) - g(t)\right|^2 \mathrm{d}t = \mathrm{O}_P\left(n^{-2d/(2d+1)}\right). \tag{3}$$

The main aim of this paper is to estimate the Skorochod's distance $d_S(\widehat{g}^*, g)$. Let us recall that

$$d_S(x, y) = \inf_{\lambda \in \Lambda} \Big\{ \sup_{0 \leqslant t \leqslant 1} \left|x(t) - y(\lambda(t))\right| + \sup_{0 \leqslant t \leqslant 1} \left|t - \lambda(t)\right| \Big\},$$

for $x, y \in D[0, 1]$, where Λ is a class of continuous monotonically increasing functions such that $\lambda(0) = 0$ and $\lambda(1) = 1$ (see [3] for more information). We do not assume that the jump point τ or its size Δ are estimated in some particular way.

Theorem 1. *Assume that the kernels K, $K_+(\cdot, q)$ and $K_-(\cdot, q)$ are continuously differentiable. Assume that the function $f(t), t \in [0, 1]$ satisfies Hölder condition with exponent one and assume $E|\xi_1|^p < \infty$ for some $p > 2$. Then for the estimator $\widehat{g}^*$ given by (2) it holds*

$$Ed_S(\widehat{g}^*, g) \leqslant E|\widehat{\tau} - \tau| + c_1 E|\widehat{\Delta} - \Delta| + c_2\Big(h + \frac{1}{h\sqrt{n}}\Big) + c_3 h^{-1} E(\widehat{\Delta}|\tau - \widehat{\tau}|),$$

where $c_1, c_2, c_3 > 0$ are constants depending on kernels k and Hölder constant of the function f.

2. Proof of Theorem 1

For a $x \in D[0, 1]$ we denote

$$||x||_\infty = \sup_{0 \leqslant t \leqslant 1} |x(t)|.$$

By the definition of the metric d_S we have

$$Ed_S(\widehat{g}^*, g) \leqslant E||\widehat{g}^* - g \circ \lambda||_\infty + E \sup_{0 \leqslant t \leqslant 1} |t - \lambda(t)|$$

for any random function $\lambda \in \Lambda$. Fix the function $\lambda \in \Lambda$ which is obtained by linearly interpolating the points $(0, 0)$, $(\widehat{\tau}, \tau)$, $(1, 1)$. Hence

$$\max_{t \in [0,1]} |t - \lambda(t)| = |\widehat{\tau} - \tau|. \tag{4}$$

Therefore we need to estimate

$$I = E\|\widehat{g}^* - g\circ\lambda\|_\infty \leqslant \|I_1\|_\infty + \|I_2\|_\infty + \|I_3\|_\infty + \|I_4\|_\infty,$$

where

$$I_1(t) = \sum_{i=1}^{n} f(t_i) \int\limits_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u - f(\lambda(t)),$$

$$I_2(t) = \widehat{\Delta}1_{[\widehat{\tau},1]}(t) - \Delta 1_{[\tau,1]}(\lambda(t)),$$

$$I_3(t) = \sum_{i=1}^{n}(\Delta 1_{[\tau,1]}(t_i) - \widehat{\Delta}1_{[\widehat{\tau},1]}(t_i)) \int\limits_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u,$$

$$I_4(t) = \sum_{j=1}^{n} \varepsilon_i \int\limits_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u,$$

and where $K_h^*(\cdot) = h^{-1}K^*(\cdot/h)$. By the definition of λ it follows that $t \geqslant \widehat{\tau}$ if and only if $\lambda(t) \geqslant \tau$. Hence

$$\|I_2\|_\infty = |\widehat{\Delta} - \Delta|. \tag{5}$$

To estimate $\|I_1\|_\infty$, note that

$$\sum_{i=1}^{n} \int\limits_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u = \int\limits_0^1 K_h^*(t-u)\mathrm{d}u = 1.$$

So

$$I_1(t) = \sum_{i=1}^{n}[f(t_i) - f(t)] \int\limits_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u + f(t) - f(\lambda(t)).$$

Evidently

$$|f(t) - f(\lambda(t))| \leqslant \|f\|_1 \sup_{0\leqslant t\leqslant 1} |t - \lambda(t)| = \|f\|_1 |\widehat{\tau} - \tau|,$$

where $\|f\|_1 = \sup_{s\neq t} |f(s) - f(t)|/|t-s|$. Noting that $\int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u \neq 0$ only if $[s_{i-1}, s_i] \cap A(t,h) \neq \varnothing$, where

$$A(t,h) = \begin{cases} [t-h, t+h], & \text{if } h \leqslant t \leqslant 1-h, \\ [t, t+h], & \text{if } 0 \leqslant t < h, \\ [t-h, t], & \text{if } 1-h < t \leqslant 1. \end{cases}$$

We have

$$\sum_{i=1}^{n}[f(t_i) - f(t)] \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u$$
$$\leqslant ||f||_1 \sum_{i=1}^{n} |t_i - t| \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u \leqslant ||f||_1(h + 1/n).$$

Hence

$$||I_1||_\infty \leqslant ||f||_1\Big(|\widehat{\tau} - \tau| + h + \frac{1}{n}\Big). \tag{6}$$

Now we estimate I_3

$$I_3(t) = \sum_{i=1}^{n}(\Delta 1_{[\tau,1]}(t_i) - \widehat{\Delta}1_{[\widehat{\tau},1]}(t_i)) \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u =$$
$$= \sum_{i=1}^{n}(\Delta 1_{[\tau,1]}(t_i) - \widehat{\Delta}1_{[\tau,1]}(t_i) + \widehat{\Delta}1_{[\tau,1]}(t_i) - \widehat{\Delta}1_{[\widehat{\tau},1]}(t_i) \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u$$
$$= \sum_{i=1}^{n}(\Delta - \widehat{\Delta})1_{[\tau,1]}(t_i) + \widehat{\Delta}1_{[\widehat{\tau},\tau]}(t_i) \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u.$$

We have

$$\sum_{i=1}^{n}(\Delta - \widehat{\Delta})1_{[\tau,1]}(t_i)) \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u$$
$$\leqslant |\Delta - \widehat{\Delta}|\widehat{\Delta} \sum_{i=1}^{n} 1_{[\widehat{\tau},\tau]}(t_i) \int_{s_{i-1}}^{s_i} K_h^*(t-u)\mathrm{d}u \leqslant \widehat{\Delta}\frac{|\tau - \widehat{\tau}|}{h}.$$

Hence

$$||I_3||_\infty \leqslant |\Delta - \widehat{\Delta}| + \widehat{\Delta}|\tau - \widehat{\tau}|h^{-1}.$$

In order to estimate $E||I_4||_\infty$, we need some preparation.

Let $T = [0, 1)$ and let d be a pseudo metric on the set T. For $\tau > 0$, denote by $N(T, d, \tau)$ the minimal number of open d-balls of radius τ which form a covering of T. Denote further by $D = D(T)$ the d-diameter of T. The following Lemma is a corollary from Theorem 1.11 in Ledoux and Talagrand [4].

Lemma 2. *Let $X = (X_t)_{t\in[0,1]}$ be a random process in L_p such that for all $s,t \in [0,1]$*

$$E|X_s - X_t|^p \leqslant d^p(s,t),$$

where d is a pseudo metric on the set $[0,1]$. Then, if

$$\int_0^D N^{1/p}(T,d,\tau)\mathrm{d}\tau < \infty,$$

we have

$$E \sup_{s,t\in T} |X_s - X_t| \leqslant 8 \int_0^D N^{1/p}(T,d,\tau)\mathrm{d}\tau.$$

Set

$$k_i(t) = \int_{s_{i-1}}^{s_i} K^*\Big(\frac{t-u}{h}\Big)\mathrm{d}u, \quad i = 1,\dots,n; \quad t \in [0,1].$$

Let $p > 2$. By Rosenthal's inequality we have for all $s,t \in [0,1]$

$$\begin{aligned} E|I_4(t) - I_4(s)|^p &\leqslant h^{-p}E\Big|\sum_{i=1}^n \epsilon_i(k_i(t) - k_i(s))\Big|^p \\ &\leqslant ch^{-p}\Big(\sum_{i=1}^n (k_i(t) - k_i(s))^2\Big)^{p/2} + ch^{-p}E|\xi_1|^p \sum_{k=1}^n |k_i(t) - k_i(s)|^p \\ &\leqslant ch^{-p}(1 + E|\xi_1|^p)\Big(\sum_{i=1}^n (k_i(t) - k_i(s))^2\Big)^{p/2}, \end{aligned}$$

where the constant $c > 0$ depends on p only. Observing that

$$|k_i(t) - k_i(s)| \leqslant ||K^*||_1 \frac{|t-s|}{hn},$$

and

$$\sum_{i=1}^n k_i(t) = h,$$

we obtain

$$E|I_4(t) - I_4(s)|^p \leqslant c\big(1 + E|\xi_1|^p\big)|t-s|^{p/2}h^{-p}n^{-p/2}.$$

By Lemma 2

$$E||I_4||_\infty \leqslant Ch^{-1}n^{-1/2}.$$

Collecting estimates we complete the proof.

References

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Apie trūkios regresijos funkcijos branduolinio įvertinimo tolygią paklaidą

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Gautas Skorochodo erdvei atstumo tarp regresijos funkcijos, turinčios trūkio tašką, ir jos branduolinio įverčio įvertinimas.