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# On the universality of Dirichlet series with multiplicative coefficients

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Let $s = \sigma + it$ be a complex variable and, as usual, $\zeta(s)$ denote the Riemann zeta-function. S.M. Voronin in [9] discovered one more remarkable property of the function $\zeta(s)$. He observed that $\zeta(s)$ is an universal object in some sense. Note that in mathematics there are many not explicitly given universal objects, while the function $\zeta(s)$ was the first explicitly given universal mathematical object. More precisely S.M. Voronin [9] proved that for any continuous and non-vanishing on $|s| \leqslant r$ function $f(s)$ which is analytic in $|s| < r$, $0 < r < 1/4$, and any $\varepsilon > 0$ there exists a real number $\tau$ such that

$$\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon.$$

Some years ago Professor I.A. Ibragimov informed the first author that a famous numbertheorist Ju.V. Linnik after Voronin's result conjectured that all functions given by Dirichlet series and analytically continuable to the left of the absolute convergence half-plane are universal. Now the latter conjecture is called the Linnik–Ibragimov hypothesis.

Later many mathematicians improved and generalized the Voronin theorem. Among them A. Reich, S.M. Gonek, B. Bagchi, K. Matsumoto, H. Mishou, the first author and others.

In [1]–[6] the universality of zeta-functions

$$Z(s)=\sum_{m=1}^{\infty}\frac{g(m)}{m^s},\quad \sigma>1,$$

with multiplicative coefficients $g(m)$, $|g(m)| \leqslant 1$, was investigated. We recall that a complex-valued arithmetic function $g(m)$ is multiplicative if $g(1) = 1$ and $g(mn) = g(m)g(n)$ for all naturals $m$ and $n$, $(m,n) = 1$. In the papers [1], [2] the case $g(p) = 1$ for any prime $p$ was considered. The papers [3] and [5], see also [6], are devoted to the universality of the function $Z(s)$ with $g(m) \in \mathcal{M}(\delta)$. Denote by $B$ a quantity bounded by a constant.

A multiplicative function $g(m)$, $|g(m)| \leqslant 1$ belongs to the class $\mathcal{M}(\delta)$ if the following three conditions are satisfied:

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1. For $x \to \infty$

$$\sum_{m \leqslant x} g(m) = M(g)x + R(x),$$

where $R(x) = Bx^{\delta}$ with $0 \leqslant \delta < 1$;
2. There exists a constant $c_1 > 0$ such that

$$\sum_{\alpha=1}^{\infty} \frac{|g(p^{\alpha})|}{2^{\alpha\sigma_0}} \leqslant c_1 < 1, \quad \sigma_0 = \frac{2+\delta}{3};$$

3. There exists a constant $c_2 > 0$ such that

$$\inf_{p} |g(p)| \geqslant c_2 > 0.$$

The aim of this paper is to obtain the universality of the function $Z(s)$ for functions $g(m)$ from a class different from $\mathcal{M}(\delta)$.

We say that a multiplicative function $g(m)$ belongs to the class $\mathcal{M}_{\beta,\theta}(C_1, C_2)$ if the following conditions are satistied:

1. There exists a constant $C_1 > 0$ such that $|g(m)| \leqslant C_1$ for all naturals $m$;
2. The function $Z(s)$ has an analytic continuation of finite order to the half-plane $\sigma \geqslant \beta$ with some $\beta \geqslant 1/2$, except, maybe, for a simple pole at $s = 1$;
3. For $\sigma \geqslant \beta$ the estimate

$$\int_0^T |Z(\sigma + it)|^2 \, dt = BT, \quad T \to \infty$$

is valid;
4. There exists a constant $\theta$ such that

$$\sum_{p \leqslant x} |g(p)|^2 = \frac{\theta x}{\log x}(1 + (1)), \quad x \to \infty;$$

5. There exists a constant $C_2$, $0 < C_2 < 1$, such that

$$\sum_{\alpha=1}^{\infty} \frac{|g(p^{\alpha})|}{p^{\beta\alpha}} \leqslant C_2$$

for all primes $p$.

Denote by $D$ the strip $\{s \in \mathbb{C}\colon \beta < \sigma < 1\}$, and let, for brevity,

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T] : \ldots\},$$

where meas$\{\cdot\}$ denotes the Lebesgue measure and in place of dots a condition satisfied by $\tau$ is to be written.

**Theorem.** *Suppose that a multiplicative function* $g(m) \in \mathcal{M}_{\beta,\theta}(C_1, C_2)$. *Let* $K$ *be a compact subset of the strip* $D$ *with connected complement, and let* $f(s)$ *be a non-vanishing continuous function on* $K$ *which is analytic in the interior of* $K$. *Then for every* $\varepsilon > 0$

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K}\big|Z(s+i\tau) - f(s)\big| < \varepsilon\Big) > 0.$$

We begin the proof of the theorem with a limit theorem for the function $Z(s)$. Let $G$ be a region on $\mathbb{C}$, and let $H(G)$ stand for the space of analytic on $G$ functions equipped with the topology of uniform convergence on compacta. We need a limit theorem in the sense of the weak convergence of probability measures for the function $Z(s)$. Let, for $N > 0$, $D_N = \{s \in \mathbb{C}: \beta < \sigma < 1\}, |t| < N\}$, and let $\mathcal{B}(S)$ stand for the class of Borel sets of the space $S$. Define on $\big(H(D_N), \mathcal{B}(H(D_N))\big)$ the probability measure

$$P_T(A) = \nu_T\big(Z(s+i\tau) \in A\big).$$

For the identification of the limit measure of $P_T$ we use the following topological structure. Denote by $\gamma$ the unit circle on $\mathbb{C}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime $p$. The infinitedimensional torus $\Omega$ is a compact topological Abelian group. Therefore there exists the probability Haar measure $m_H$ on $(\Omega, \mathcal{B}(\Omega))$, and we obtain the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space $\gamma_p$,and define an $H(D_N)$-valued random element $Z(s,\omega)$ on $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$ by the formula

$$Z(s,\omega) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)\omega^\alpha(p)}{p^{\alpha s}}\right), \quad s \in D_N,\ \omega \in \Omega.$$

Denote by $P_Z$ the distribution of the random element $Z(s,\omega)$.

**Lemma 1.** *The probability measure* $P_T$ *converges weakly to* $P_Z$ *as* $T \to \infty$.

*Proof.* A theorem of such kind for $g(m) \in \mathcal{M}(\delta)$ in the space $H(\Delta)$, $\Delta = \{s \in \mathbb{C}: \sigma_0 < \sigma < 1\}$ was proved in [6], Theorem 9.11. In our case the lemma in the space $H(D)$ can be obtained in a completely similar manner to that of [6]. From this the lemma follows immediately in the space $H(D_N)$.

Now let, for $a_p \in \gamma$, $s \in D_N$,

$$f_p(s, a_p) = \log\left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)a_p^\alpha}{p^{\alpha s}}\right),$$

where, for $|z| < 1$,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots .$$

In virtue of the condition $5^0$ of the class $\mathcal{M}_{\beta,\theta}(C_1, C_2)$ we have that the definition of $f_p(s, a_p)$ is correct.

**Lemma 2.** *Suppose that a multiplicative function $g(m) \in \mathcal{M}_{\beta,\theta}(C_1, C_2)$. Then the set of all convergent series*

$$\sum_p f_p(s, a_p)$$

*is dense in $H(D_N)$.*

*Proof.* When $C_1 = 1$, the lemma is proved in[7]. The general case is considered similarly to that of [7].

Now we will find the support of the measure $P_Z$. Let

$$S_N = \{f \in H(D_N)\colon f(s) \neq 0 \quad \text{for} \quad s \in D_N \quad \text{or} \quad f(s) \equiv 0\}.$$

**Lemma 3.** *The support $S(Z)$ of the measure $P_Z$ is the set $S_N$.*

*Proof.* We have by the definition that $\{\omega(p)\}$ is a sequence of independent random variables on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. The support of each $\omega(p)$ is the unit circle $\gamma$. Therefore

$$\left\{\log\left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)\omega^\alpha(p)}{p^{\alpha s}}\right)\right\}$$

is a sequence of independent $H(D_N)$-valued random elements, and the set

$$\left\{f \in H(D_N)\colon f(s) = \log\left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)\omega^\alpha(p)}{p^{\alpha s}}\right),\ a \in \gamma\right\}$$

is the support of each element. Therefore by Theorem 1.7.10 of [6] the support of the $H(D_N)$-valued random element

$$\log Z(s, \omega) = \sum_p \log\left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)\omega^\alpha(p)}{p^{\alpha s}}\right)$$

is the closure of the set of all convergent series

$$\sum_p f_p(s, a_p)$$

in the notation of Lemma 2. By Lemma 2 the latter set is dense in $H(D_N)$. Hence and from the continuity of the exponent function we obtain that the support of $Z(s,\omega)$ contains the set $S_N \setminus \{0\}$. However, the support is a closed set. By the Hurwitz theorem on the uniform convergence of sequences of analytic functions [8] we find that $\overline{S_N \setminus \{0\}} = S_N$. Therefore

$$S(Z) \supseteq S_N. \tag{1}$$

On the other hand, by the definition of the class $\mathcal{M}_{\beta,\theta}(C_1, C_2)$

$$1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)\omega^\alpha(p)}{p^{\alpha s}} \neq 0, \quad s \in D_N,\ \omega \in \Omega,$$

for each prime $p$. This means that $Z(s,\omega)$ is an almost surely convergent product of non-vanishing factors. Hence by the Hurwitz theorem again we have that $Z(s,\omega) \in S_N$ almost surely, i.e. $S(Z) \subseteq S_N$. From this and (1) the lemma follows.

*Proof of theorem.* Let $K$ be a compact subset of $D$ with connected complement. Then there exists a number $N > 0$ such that $K \subset D_N$.

First let $f(s)$ have the non-vanishing analytic continuation to $H(D_N)$. Consider the set of functions

$$G = \Big\{ g \in H(D_N)\colon \sup_{s\in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Clearly, the set $G$ is open. Therefore Lemma 1 and properties of the weak convergence of probability measures yield

$$\liminf_{T\to\infty} \nu_T\Big( \sup_{s\in K} \big|Z(s+i\tau) - f(s)\big| < \varepsilon \Big) \geqslant P_Z(G). \tag{2}$$

In this case $f \in S_N$, therefore by Lemma 3 $f \in S(Z)$. However, the set $G$ is a neighbourhood of $f$, hence $P_Z(G) > 0$. From this and (2) in the considered case the theorem follows.

Now let $f(s)$ be as in the theorem. Then by the Mergelyan theorem on the approximation of analytic functions by polynomials, see, for example, [10], there exists a polynomial $p(s)$, $p(s) \neq 0$ on $K$, such that

$$\sup_{s\in K} \big|f(s) - p(s)\big| < \frac{\varepsilon}{4}. \tag{3}$$

The polynomial $p(s)$ has only finitely many zeros. Therefore there exists a region $G_1$ such that $K \subset G_1$ and $p(s) \neq 0$ on $G_1$. Now we choose $\log p(s)$ to be analytic in the interior of $G_1$. By the Mergelyan theorem again there exists a polynomial $g(s)$ such that

$$\sup_{s\in K} \left|p(s) - e^{q(s)}\right| < \frac{\varepsilon}{4}.$$

Now this and (3) give

$$\sup_{s\in K} \left|f(s) - e^{q(s)}\right| < \frac{\varepsilon}{2}. \tag{4}$$

However, $e^{q(s)} \neq 0$. Therefore by the considered case

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K} \left|Z(s+i\tau) - e^{q(s)}\right| < \frac{\varepsilon}{4}\right) > 0.$$

From this and (4) the assertion of the theorem follows in general case.

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# Apie Dirichlet eilučių su multiplikatyviaisiais koeficientais universalumą

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Įrodyta, kad funkcija, srityje $\sigma > 1$ apibrėžiama Dirichlet eilute su multiplikatyviais koeficientais iš klasės $\mathcal{M}_{\beta,\theta}(C_1, C_2)$, yra universali: jos postūmiais tolygiai kompaktinėse aibėse galima aproksimuoti analizines funkcijas.