## A note on the value-distribution of the periodic zeta-function

Audrius KAČĖNAS\* (VU), Antanas LAURINČIKAS (VU, ŠU) e-mail: antanas.laurincikas@maf.vu.lt

Let  $\mathbb{Z}$  denote the set of all integer numbers, and  $\mathfrak{A} = \{a_m, m \in \mathbb{Z}\}$  be a sequence of complex numbers with period k > 0. In [1] the periodic zeta-function  $\zeta(s; \mathfrak{A})$ ,  $s = \sigma + it$ , was introduced and studied. For  $\sigma > 1$ , the function  $\zeta(s; \mathfrak{A})$  is defined by

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

If  $\mathfrak{A} = \{1\}$  and k = 1, then the function  $\zeta(s; \mathfrak{A})$  reduces to the classical Riemann zeta-function  $\zeta(s)$ . The equality

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} = \sum_{q=1}^{k} a_q \sum_{m=0}^{\infty} \frac{1}{(mk+q)^s}$$
$$= \frac{1}{k^s} \sum_{q=1}^{k} a_q \zeta\left(s, \frac{q}{k}\right),$$

which is valid for  $\sigma > 1$ , and  $\zeta(s, \alpha)$  denotes the Hurwitz zeta-function, gives the analytic continuation of the function  $\zeta(s; \mathfrak{A})$  into the entire s-plane where it is regular with the possible exception of a simple pole at s = 1 with the residue

$$a = \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

In [3] the mean square of  $\zeta(s;\mathfrak{A})$  in the strip  $\frac{1}{2} \leqslant \sigma \leqslant 1$  was studied and some limit theorems in the sense of the weak convergence of probability measures were proved.

Denote by  $meas\{A\}$  the Lebesgue measure of the set A, and let, for T>0,

$$u_T(\ldots) = \frac{1}{T} meas\{\tau \in [0,T],\ldots\},$$

where instead of dots a condition satisfied by  $\tau$  is to be written. Let  $\mathcal{B}(S)$  stand for the class of Borel sets of the space S. Denote by H(G) the space of analytic on G functions

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equipped with the topology of uniform convergence on compacta. Moreover, let  $\gamma$  be the unit circle on the complex plane,

$$\Omega = \prod_{p} \gamma_{p},$$

where  $\gamma_p = \gamma$  for all primes p. Denote by  $\omega(p)$  the projection of  $\omega \in \Omega$  to the coordinate space  $\gamma_p$ , and let, for natural m,

$$\omega(m) = \prod_{p^{\alpha} || m} \omega^{\alpha}(p).$$

Let  $\mathbb{C}$  be the complex plane, and  $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ . On the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ , where  $m_H$  denotes the Haar measure on  $(\Omega, \mathcal{B}(\Omega))$ , define an H(D)-valued random element  $\zeta(s, \omega; \mathfrak{A})$  by the formula

$$\zeta(s,\omega;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D.$$

**Theorem A** [3]. The probability measure

$$\nu_T(\zeta(s+i\tau;\mathfrak{A})\in A), \quad A\in\mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element  $\zeta(s,\omega;\mathfrak{A})$  as  $T\to\infty$ .

The aim of this note is to obtain a similar limit theorem in the space of functions defined in the half-plane  $D_1 = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$ .

**Theorem 1.** Suppose that a = 0. Then the probability measure

$$\nu_T(\zeta(s+i\tau;\mathfrak{A})\in A), \quad A\in\mathcal{B}(H(D_1)),$$

converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D_1,$$

as  $T \to \infty$ .

Let  $\mathbb{C}_{\infty}=\mathbb{C}\cup\{\infty\}$  be the Riemann sphere, and let  $d(s_1,s_2)$  be a metric on  $\mathbb{C}_{\infty}$  given by the formulae

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2} \sqrt{1 + |s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0.$$

Here  $s, s_1, s_2 \in \mathbb{C}$ . This metric is compatible with the topology of  $\mathbb{C}_{\infty}$ . Denote by  $M(D_1)$  the space of meromorphic on  $D_1$  functions  $f: D_1 \to (\mathbb{C}_{\infty}, d)$  equipped with the topology of uniform convergence on compacta. In this topology, a sequence  $\{f_n, f_n \in M(D_1)\}$  converges to the function  $f \in M(D_1)$  if

$$d(f_n(s), f(s)) \to 0$$

as  $n \to \infty$  uniformly in s on compact subsets of  $D_1$ .

**Theorem 2.** Suppose that  $a \neq 0$ . Then the probability measure

$$\nu_T(\zeta(s+i\tau;\mathfrak{A})\in A), \quad A\in\mathcal{B}(M(D_1)),$$

converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D_1.$$

Proof of Theorem 1 completely coincides with that of Theorem A.

Proof of Theorem 2. Let

$$\zeta_1(s;\mathfrak{A}) = (1 - 2^{1-s})\zeta(s;\mathfrak{A}).$$

Then the function  $\zeta_1(s;\mathfrak{A})$  is analytic in  $D_1$ , and, for  $\sigma > 1$ , it is given by an absolutely convergent Dirichlet series. Let B be a factor bounded by a constant. Since, for  $\sigma > \frac{1}{2}$ , [3]

$$\int_{1}^{T} \left| \zeta(\sigma + it; \mathfrak{A}) \right|^{2} dt = BT, \quad T \to \infty,$$

clearly,

$$\int_{1}^{T} |\zeta_{1}(\sigma + it; \mathfrak{A})|^{2} dt = BT, \quad T \to \infty.$$

Moreover,  $\zeta_1(s;\mathfrak{A})$  is a function of finite order. Therefore, the probability measure

$$Q_T(A) = \nu_T(\zeta_1(s+i\tau;\mathfrak{A}) \in A), \quad A \in \mathcal{B}(H(D_1))$$

converges weakly to the distribution of the random element

$$(1-2^{1-s}\omega(2))\sum_{m=1}^{\infty}\frac{a_m\omega(m)}{m^s}, \quad \omega\in\Omega, \quad s\in D_1,$$

as  $T \to \infty$ . This can be obtained similarly to the proof of Theorem A and Theorem 1. The function

$$p(s) = 1 - 2^{1-s}$$

is a Dirichlet polynomial. Therefore, the probability measure

$$Q_{T,p}(A) = \nu_T(p(s+i\tau) \in A), \quad A \in \mathcal{B}(H(D_1)),$$

converges weakly to the distribution of the random element.

$$1-2^{1-s}\omega(2), \quad \omega \in \Omega, \quad s \in D_1,$$

as  $T \to \infty$  [4]. The weak convergence of the probability measures  $Q_T$  and  $Q_{T,p}$  implies the weak convergence of the probability measure

$$P_T(A) = \nu_T\Big(\big(\zeta_1(s+i\tau;\mathfrak{A}), 1-2^{1-s-i\tau}\big) \in A\Big), \quad A \in \mathcal{B}\big(H^2(D_1)\big),$$

to the distribution of the random element

$$\left(\left(1 - 2^{1-s}\omega(2)\right)\sum_{m=1}^{\infty} \frac{a_m\omega(m)}{m^s}, \quad 1 - 2^{1-s}\omega(2)\right), \quad \omega \in \Omega, \quad s \in D_1$$
 (1)

as to  $T \to \infty$ . The details are the same as, for example, in [5].

Let the function  $h: H^2(D_1) \to M(D_1)$  be given by the formula

$$h(f_1, f_2) = \frac{f_1}{f_2}, \quad f_1, f_2 \in H(D_1).$$

Since the metric d satisfies the equality

$$d(f_1,f_2)=d\left(\frac{1}{f_1},\frac{1}{f_2}\right),\,$$

the function h is continuous. Hence, by Theorem 5.1 from [2], we obtain that the measure  $P_T h^{-1}$  converges weakly to  $Ph^{-1}$  as  $T \to \infty$ , where P is the distribution of the random element (1). Thus, in view of the definition of the function h, Theorem 2 follows.

## References

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<sup>[2]</sup> P. Billingsley, Convergence of Probability Measures, John Wiley and Sons, New York (1968).

<sup>[3]</sup> A. Kačenas, A. Laurinčikas, On the periodic zeta-function, Liet. Matem. Rink. (to appear).

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## Pastaba apie periodinės dzeta funkcijos reikšmių pasiskirstymą

## A. Kačėnas, A. Laurinčikas

Straipsnyje įrodytos ribinės teoremos periodinei dzeta funkcijai analizinių ir meromorfinių funkcijų erdvėse.