Liet. matem. rink., T. 40, spec. nr., 43–48

On the mean square of the Lerch zeta-function with respect to the parameter

Antanas LAURINČIKAS* (VU, ŠU)
e-mail: antanas.laurincikas@maf.vu.lt

The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, for $\sigma > 1$, is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. Here λ and $\alpha > 0$ are fixed real parameters. If λ is an integer, then the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$. When λ is not an integer, then the function $L(\lambda, \alpha, s)$ is analytically continuable to an entire function.

Let

$$\widetilde{L}(\lambda, \alpha, s) = L(\lambda, \alpha, s) - \alpha^{-s},$$

and

$$I(s, \lambda) = \int_0^1 \left|\widetilde{L}(\lambda, \alpha, s)\right|^2 d\alpha.$$

The aim of this paper is to obtain the formulae for $I(\frac{1}{2} + it, \lambda)$ and $I(1 + it, \lambda)$ by using a formula for $I(\sigma + it, \lambda)$, $\frac{1}{2} < \sigma < 1$.

Define the function $\widetilde{\zeta}(\lambda, s)$, for $\sigma > 1$, by

$$\widetilde{\zeta}(\lambda, s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and by analytic continuation elsewhere. Let, as usual, $\Gamma(s)$ stand for the Euler gamma-function, t_0 be a positive number, and B denote a number bounded by a constant.

Theorem A. *Let $\frac{1}{2} < \sigma < 1$ be fixed and $t \geqslant t_0$. Then for any real λ*

$$I(\sigma + it, \lambda) = \frac{1}{2\sigma - 1} + 2\Gamma(2\sigma - 1)\Re\left(\widetilde{\zeta}(\lambda, 2\sigma - 1)\frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)}\right)$$

*Partially supported by Lithuanian State Studies and Science Foundation.

$$- 2\Re \frac{1}{1-\sigma+it}\left(e^{-2\pi i\lambda}\widetilde{\zeta}(\lambda, \sigma+it) - 1\right) + Bt^{-1}.$$

Proof of Theorem A can be found in [1], [2].

Let γ be the Euler constant.

Theorem 1. *Let $t \geqslant t_0$. Then for any real λ*

$$I\left(\frac{1}{2}+it, \lambda\right) = \gamma + 2\Re\left(\widetilde{\zeta}'(\lambda, 0) - \widetilde{\zeta}(\lambda, 0)\frac{\Gamma'\left(\frac{1}{2}+it\right)}{\Gamma\left(\frac{1}{2}+it\right)}\right)$$
$$- 2\Re \frac{e^{-2\pi i\lambda}\widetilde{\zeta}\left(\lambda, \frac{1}{2}+it\right) - 1}{\frac{1}{2}+it} + Bt^{-1}.$$

Theorem 2. *Let $t \geqslant t_0$ and $0 < \lambda < 1$. Then*

$$I(1+it, \lambda) = 1 + \pi(1-2\lambda)t^{-1} - 2\Re\frac{1}{it}\left(e^{-2\pi i\lambda}\widetilde{\zeta}(\lambda, 1+it) - 1\right)$$
$$- 2\Re\frac{1}{it}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{m(m+1)^{1+it}}.$$

For the proof of Theorem 1 we will apply some results on the summation of divergent series. We recall the summation in the Abel sense. Suppose that the series

$$\sum_{m=0}^{\infty} a_m x^m,$$

for $0 < x < 1$, converges and has a sum $f(x)$. If

$$\lim_{x \to 1-0} f(x) = A,$$

then the series

$$\sum_{m=0}^{\infty} a_m$$

is called summable in the Abel sense, and A is its generalized sum.

Lemma 1. *Let $0 < \lambda < 1$. Then for all s the series*

$$\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{m^s}$$

is summable in the Abel sense and has the generalized sum $\widetilde{\zeta}(\lambda, s)$.

Proof. First of all we note that $\tilde{\zeta}(\lambda, s)$ is an entire function. This can be obtained similarly to the case of the Lerch zeta-function discussed in Section 2.2 of [1].

Let $0 < x < 1$. Then, for $\sigma > 1$,

$$\sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}}{m^s} x^m = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} x^m}{\Gamma(s)} \int_0^{\infty} e^{-mu} u^{s-1}\, du$$
$$= \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{m=1}^{\infty} e^{2\pi i\lambda m} x^m e^{-mu}\, du$$
$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x e^{2\pi i\lambda} e^{-u} u^{s-1}}{1 - x e^{2\pi i\lambda} e^{-u}}\, du.$$

By analytic continuation this remains true for all s. Similarly as in [1], Section 2.2, hence we find that

$$\sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} x^m}{m^s} = \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} \int_L \frac{x e^{2\pi i\lambda} e^{-z} z^{s-1}}{1 - x e^{2\pi i\lambda} e^{-z}}\, dz. \tag{1}$$

Here the contour L consists of the part of the real axis from ∞ to 0, encloses the point $z = 0$ and returns to ∞. The points $\log x + 2\pi i\lambda + 2\pi i k$, k is an integer, do not belong to this contour and its interior.

The right-hand side of the equality (1) converges uniformly to a limit as $x \to 1 - 0$ on every bounded part of s-plane non-containing the points $s = n$ where n is a natural number. By the principle of analytic continuation this limit is $\tilde{\zeta}(\lambda, s)$. The case $s = n$ is trivial. Therefore, for all s,

$$\lim_{x \to 1-0} \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} x^m}{m^s} = \tilde{\zeta}(\lambda, s),$$

and the lemma is proved.

In view of Lemma 1 the series

$$\sum_{m=1}^{\infty} \cos 2\pi\lambda m, \quad 0 < \lambda < 1, \tag{2}$$

is summable in the Abel sense, and it remains to find its generalized sum.

Lemma 2. *The generalized sum of the series* (2) *is* $-\frac{1}{2}$.

Proof. Let $0 < \theta < 2\pi$ and $0 < x < 1$. It is convienent instead of (2) to consider the series

$$\frac{1}{2} + \sum_{m=1}^{\infty} \cos 2\pi\lambda m. \tag{3}$$

It is not difficult to see that

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1 + 2\sum_{m=1}^{\infty} x^m \cos\theta m. \tag{4}$$

Really, multiplying the right-hand side of (4) by $1 - 2x\cos\theta + x^2$, we obtain

$$1 - 2x\cos\theta + x^2 + 2\sum_{m=1}^{\infty} x^m \cos\theta m$$
$$- 2\sum_{m=1}^{\infty} x^{m+1} 2\cos\theta m \cos\theta + 2\sum_{m=1}^{\infty} x^{m+2}\cos\theta m.$$

Since

$$2\cos\theta m\cos\theta = \cos(m+1)\theta + \cos(m-1)\theta,$$

the last expression is equal to

$$1 - 2x\cos\theta + x^2 + 2x\cos\theta + 2\sum_{m=2}^{\infty} x^m\cos\theta m - 2\sum_{m=2}^{\infty} x^m \cos\theta m$$
$$- 2x^2 - 2x^2\sum_{m=1}^{\infty} x^m\cos m\theta + 2x^2\sum_{m=1}^{\infty} x^m \cos m\theta = 1 - x^2.$$

Let $\theta = 2\pi\lambda$. Then taking a limit as $x \to 1 - 0$ in (4), we find that the generalized sum of (3) is 0.

Proof of Theorem 1. We will deduce Theorem 1 from Theorem A taking a limit as $\sigma \to \frac{1}{2} + 0$. First let λ be a noninteger. Then we have as $\sigma \to \frac{1}{2} + 0$

$$\tilde{\zeta}(\lambda, 2\sigma - 1) = \tilde{\zeta}(\lambda, 0) + (2\sigma - 1)\tilde{\zeta}'(\lambda, 0) + o(2\sigma - 1),$$
$$\Gamma(1 - \sigma + it) = \Gamma(\sigma + it) - (2\sigma - 1)\Gamma'(\sigma + it) + o(2\sigma - 1).$$

Moreover, it is well known that

$$\Gamma(2\sigma - 1) = \frac{1}{2\sigma - 1} - \gamma + B(2\sigma - 1).$$

Consequently, the right-hand side of the equality of Theorem A is

$$\frac{1}{2\sigma-1}+\frac{2}{2\sigma-1}\Re\widetilde{\zeta}_\lambda(0)-2\gamma\widetilde{\zeta}_\lambda(0)+2\Re\Big(\widetilde{\zeta}'(\lambda,0)-\widetilde{\zeta}(\lambda,0)\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)}\Big)$$
$$-2\Re\frac{1}{1-\sigma+it}\big(e^{-2\pi i\lambda}\widetilde{\zeta}(\lambda,\sigma+it)-1\big)+Bt^{-1}+o(1) \qquad (5)$$

as $\sigma\to\frac{1}{2}+0$. By Lemmas 1 and 2 we have $\Re\widetilde{\zeta}(\lambda,0)=-\frac{1}{2}$. This and (5) prove the theorem.

When λ is an integer, then $\widetilde{\zeta}(\lambda,s)=\zeta(s)$ and the assertion of the theorem is obtained similarly using the equality $\zeta(0)=-\frac{1}{2}$.

Proof of Theorem 2. Let $F(a,b;c;z)$ denote the hypergeometric function. In [2] it was obtained that, for $0<\Re u<1$, $0<\Re v<1$,

$$\int_0^1 \widetilde{L}(\lambda,\alpha,u)\widetilde{L}(-\lambda,\alpha,v)\mathrm{d}\alpha=\frac{1}{u+v-1}+\Gamma(u+v-1)$$
$$\times\left(\widetilde{\zeta}(\lambda,u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)}+\widetilde{\zeta}(-\lambda,u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right)$$
$$-\frac{1}{1-v}\big(e^{-2\pi i\lambda}\widetilde{\zeta}(\lambda,u)-1\big)-\frac{1}{1-u}\big(e^{2\pi i\lambda}\widetilde{\zeta}(-\lambda,u)-1\big)$$
$$-\frac{u}{(2-v)(1-v)}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+1)^{u+1}}F\left(u+1,1;3-v;\frac{1}{m+1}\right)$$
$$-\frac{v}{(2-u)(1-u)}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+1)^{v+1}}F\left(v+1,1;3-u;\frac{1}{m+1}\right). \qquad (6)$$

Since, for $|z|<1$,

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)$$

and $F\big(0,b;c;\frac{1}{m+1}\big)=0$, we have that the series in (6) as $\sigma\to1-0$ are equal to

$$-2\Re\frac{1}{it}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{m(m+1)^{1+it}}.$$

It is easily seen that the Fourier expansion of the function $\pi(1-2\lambda)$ is

$$\sum_{m=1}^{\infty}\frac{\sin 2\pi m\lambda}{m}.$$

Hence, by Lemma 1,

$$\pi(1-2\lambda) = 2\Re\frac{1}{i}\widetilde{\zeta}(\lambda, 1).$$

Thus the theorem follows if we take $\sigma \to 1-0$ in Theorem A and in the equality (6).

References

[1] A. Garunkštis, A. Kačėnas, A. Laurinčikas, *The Lerch zeta-function*, Vilniaus universitetas, Matematikos ir informatikos fakultetas (2000) (in Lithuanian).

[2] A. Laurinčikas, The mean square of the Lerch zeta-function with respect to the parameter α, *Nonlinear Analysis: Maodeling and Control* (2000) (submitted).

Apie Lercho dzeta funkcijos kvadrato vidurkį parametro atžvilgiu

A. Laurinčikas

Gautos formulės Lercho dzeta funkcijos kvadrato vidurkiui parametro α atžvilgiu atvejais $\sigma = \frac{1}{2}$ ir $\sigma = 1$.