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# On the limits for distributions of additive functions

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Let $\{f_x, x \geqslant 2\}$ be a set of additive functions and

$$\nu_x\big(f_x(m) < u\big) = [x]^{-1}\#\{m \leqslant x:\ f_x(m) < u\}. \tag{1}$$

The following assertion about weak convergence of the distribution functions $\nu_x\big(f_x(m) < u\big)$ was proved in [1], [2].

**Theorem 1.** *Let $\{f_x, x \geqslant 2\}$ be a set of strongly additive functions and $f_x(p) \in \{0, 1\}$ for each prime number p. The distribution functions (1) converge weakly (as $x \to \infty$) if and only if there exist finite limits*

$$\lim_{x\to\infty} \sum_{\substack{p_1\leqslant x\\ f_x(p_1)=1}} \frac{1}{p_1} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1\\ f_x(p_2)=1}} \frac{1}{p_2} \cdots \sum_{\substack{p_{l-1}\leqslant x\\ p_{l-1}\neq p_1,p_2,\ldots,p_{l-2}\\ f_x(p_{l-1})=1}} \frac{1}{p_{l-1}} \sum_{\substack{p_l\leqslant x/p_1p_2\cdots p_{l-1}\\ p_l\neq p_1,p_2,\ldots,p_{l-1}\\ f_x(p_l)=1}} \frac{1}{p_l} = g_l \tag{2}$$

*for each integer $l \geqslant 1$. Moreover the characteristic function of the limit distribution is*

$$1 + \sum_{l=1}^{\infty} \frac{g_l}{l!}\big(\mathrm{e}^{it} - 1\big)^l.$$

Denote by $\mathcal{M}$ the set of limit distributions for (1) appearing in this theorem. The problem is to characterise this set. For any distribution in $\mathcal{M}$ the factorial moments $g_l$ are finite and have peculiar expressions (2). Moreover, from (2) we get

$$g_l \leqslant g_{l-k}g_k \tag{3}$$

for $k = 1, 2, \ldots, l-1$ and $l = 2, 3, \ldots$

There are two possibilities. Namely, the limit distribution has infinite support if $g_l > 0$ for all natural $l$, or it has finite support if $g_m = 0$ for some natural $m$. In the second case we have that $g_l = 0$ for all $l \geqslant m$ according to (3). In this article we shall give the description of limit distributions with supports $\{0, 1\}$ or $\{0, 1, 2\}$. Similarly can be found limit distributions with support $\{0, 1, \ldots, m\}$, but the authors prefer not to frighten readers with cumbrous expressions.

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The set of prime numbers we denote by $\mathcal{P}$. The symbol * over the sign of sum means that the sum is taken over primes $p$ with the condition $f_x(p) = 1$.

**Theorem 2.** *The distribution $F$ with support $\{0,1\}$ belongs to $\mathcal{M}$ if and only if the characteristic function of $F$ has the form*

$$\varphi(t) = 1 + g(\mathrm{e}^{it} - 1), \tag{4}$$

*where*

$$g = \frac{\varepsilon}{P} + \alpha, \quad P \in \mathcal{P}, \quad \alpha \in [0, \ln 2], \ \varepsilon \in \{0,1\}, \quad \varepsilon + \alpha > 0.$$

**Theorem 3.** *The distribution $G$ with support $\{0,1,2\}$ belongs to $\mathcal{M}$ if and only if the characteristic function of $G$ has the form*

$$\psi(t) = 1 + g_1(\mathrm{e}^{it} - 1) + \frac{g_2}{2}(\mathrm{e}^{it} - 1)^2, \tag{5}$$

*where*

$$\begin{aligned}
&g_1 = \frac{\varepsilon_1}{P_1} + \frac{\varepsilon_2}{P_2} + \alpha + \beta + \gamma, \quad P_1, P_2 \in \mathcal{P}, \quad \varepsilon_1, \varepsilon_2 \in \{0,1\},\\
&g_2 = g_1^2 - \Big(\frac{\varepsilon_1}{P_1^2} + \frac{\varepsilon_2}{P_2^2} + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma\Big) + 2\theta,\\
&\alpha, \gamma \in [0, \ln(3/2)], \quad \beta \in [0, \ln(4/3)], \quad \theta \in [\theta_1, \theta_2], \quad \alpha + \varepsilon_1 + \varepsilon_2 > 0,\\
&\theta_1 = (\beta + \ln(3/2))(\alpha + \ln 2d) + \mathrm{Li}_2(\mathrm{e}^{-\alpha}/2) - \mathrm{Li}_2(d),\\
&\theta_2 = \alpha\beta + (\alpha - \ln 3c)(\ln 2 - \beta) + \mathrm{Li}_2(c) - \mathrm{Li}_2(\mathrm{e}^{\alpha}/3),\\
&c = \min\{\mathrm{e}^{\alpha}/3,\ 1 - \mathrm{e}^{\beta}/2\}, \quad d = \max\{\mathrm{e}^{-\alpha}/2,\ 1 - 2\mathrm{e}^{-\beta}/3\},
\end{aligned}$$

*and $\mathrm{Li}_2(x)$ is the polylogarithm of the second order, i.e.,*

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \quad |x| \leqslant 1.$$

For some $\alpha, \beta$ the parameters $\theta_1$ and $\theta_2$ have more simple representation. Namely $\theta_1 = 0$ when $d = \mathrm{e}^{-\alpha}/2$ and $\theta_2 = \alpha\beta$ if $c = \mathrm{e}^{\alpha}/3$.

*Proof of Theorem 2.* I. Let $F \in \mathcal{M}$ and *supp* $F = \{0,1\}$. Theorem 1 yields

$$\lim_{x\to\infty} \sum_{p\leqslant x}^{*} \frac{1}{p} = g > 0, \qquad \lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \frac{1}{p_1} \sum_{\substack{p_2\leqslant x/p_1\\ p_2\neq p_1}}^{*} \frac{1}{p_2} = 0.$$

The last equality implies (for details see Corollary 4 in [2]) the existence of integer $D \geqslant 2$ with properties

$$\limsup_{x\to\infty} \#\{p \leqslant D:\ f_x(p) = 1\} \leqslant 1,$$

$$\lim_{x\to\infty} \sum_{D<p\leqslant\sqrt{x}}^{*} \frac{1}{p} = 0.$$

Let now $y_k$ be an arbitrary unbounded increasing sequence of real numbers. There is a subsequence $x_k$ such that the limits

$$\lim_{k\to\infty} \sum_{p\leqslant D}' \frac{1}{p} = \frac{\varepsilon}{P}, \qquad \lim_{k\to\infty} \sum_{\sqrt{x_k}<p\leqslant x_k}' \frac{1}{p} = \alpha$$

exist. Here the symbol $'$ over the sign of sum means that the sum is taken over primes $p$ with the condition $f_{x_k}(p) = 1$. Since $g = \alpha + \varepsilon/P$, the characteristic function of $F$ has the representation (4).

II. Suppose that the characteristic function of distribution $F$ is (4). Let

$$f_x(p) = \varepsilon \mathbf{1}_{\{P\}}(p) + \mathbf{1}_{(\sqrt{x},\, x^{e^{\alpha}/2}]}(p).$$

Theorem 1 guarantees that $\nu_x\big(f_x(m) < u\big)$ converge weakly to the distribution $F$. Hence $F$ belongs to $\mathcal{M}$. This proves Theorem 2.

*Proof of Theorem 3.* I. Let a distribution $G$ with support $\{0, 1, 2\}$ belongs to $\mathcal{M}$. Again by appeal to the theorem we obtain

$$\lim_{x\to\infty} \sum_{p\leqslant x}^{*} \frac{1}{p} = g_1 > 0, \qquad \lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \frac{1}{p_1} \sum_{\substack{p_2\leqslant x/p_1\\ p_2\neq p_1}}^{*} \frac{1}{p_2} = g_2 > 0,$$

$$\lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \frac{1}{p_1} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \frac{1}{p_2} \sum_{\substack{p_3\leqslant x/p_1p_2\\ p_3\neq p_1, p_2}}^{*} \frac{1}{p_3} = 0.$$

The later equality implies (see Corollary 4 in [2]) the existence of integer $D \geqslant 2$ for which

$$\limsup_{x\to\infty} \#\{p \leqslant D:\ f_x(p) = 1\} \leqslant 2,$$

$$\lim_{x\to\infty} \sum_{D<p\leqslant x^{1/3}}^{*} \frac{1}{p} = 0.$$

Any unbounded increasing sequence of real numbers contains a subsequence $x_k$ such that the limits

$$\lim_{k\to\infty} \sum_{p\leqslant D}' \frac{1}{p} = \frac{\varepsilon_1}{P_1} + \frac{\varepsilon_2}{P_2}, \qquad \lim_{k\to\infty} \sum_{x_k^{1/3}<p\leqslant x_k^{1/2}}' \frac{1}{p} = \alpha,$$

$$\lim_{k\to\infty} \sum_{x_k^{1/2}<p\leqslant x_k^{2/3}}' \frac{1}{p} = \beta, \qquad \lim_{k\to\infty} \sum_{x_k^{2/3}<p\leqslant x_k}' \frac{1}{p} = \gamma,$$

$$\lim_{k\to\infty} \sum'_{x_k^{1/3}<p_1\leqslant x_k^{1/2}} \frac{1}{p_1} \sum'_{\substack{x_k^{1/2}<p_2\leqslant x_k^{2/3}\\ p_1p_2\leqslant x_k}} \frac{1}{p_2} = \theta$$

exist. Finally, since

$$g_1 = \frac{\varepsilon_1}{P_1} + \frac{\varepsilon_2}{P_2} + \alpha + \beta + \gamma,$$

$$g_2 = \lim_{k\to\infty} \Big( \sum'_{p_1\leqslant x_k^{1/3}} \frac{1}{p_1} \sum'_{\substack{p_2\leqslant x_k/p_1\\ p_1\neq p_2}} \frac{1}{p_2} + \sum'_{x_k^{2/3}<p_1\leqslant x_k} \frac{1}{p_1} \sum'_{\substack{p_2\leqslant x_k^{1/3}\\ p_1p_2\leqslant x_k\\ p_1\neq p_2}} \frac{1}{p_2}$$

$$+ \sum'_{x_k^{1/3}<p_1\leqslant x_k^{1/2}} \frac{1}{p_1}\Big( \sum'_{\substack{p_2\leqslant x_k^{1/3}\\ p_1p_2\leqslant x_k\\ p_1\neq p_2}} \frac{1}{p_2} + \sum'_{\substack{x_k^{1/3}<p_2\leqslant x_k^{1/2}\\ p_1p_2\leqslant x_k\\ p_1\neq p_2}} \frac{1}{p_2} + \sum'_{\substack{x_k^{1/2}<p_2\leqslant x_k^{2/3}\\ p_1p_2\leqslant x_k\\ p_1\neq p_2}} \frac{1}{p_2}\Big) +$$

$$+ \sum'_{x_k^{1/2}<p_1\leqslant x_k^{2/3}} \frac{1}{p_1}\Big( \sum'_{\substack{p_2\leqslant x_k^{1/3}\\ p_1p_2\leqslant x_k\\ p_1\neq p_2}} \frac{1}{p_2} + \sum'_{\substack{x_k^{1/3}<p_2\leqslant x_k^{1/2}\\ p_1p_2\leqslant x_k\\ p_1\neq p_2}} \frac{1}{p_2}\Big)\Big)$$

$$= g_1^2 - \Big(\frac{\varepsilon_1}{P_1^2} + \frac{\varepsilon_2}{P_2^2} + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma\Big) + 2\theta,$$

we conclude that distribution $G$ has characteristic function (5). It remains only to evaluate the bounds for parameter $\theta$. Let we introduce some auxiliary functions. For $(a,b) \in W = [1/3, \mathrm{e}^{-\alpha}/2] \times [1/2, 2\mathrm{e}^{-\beta}/3]$ denote

$$\theta(a,b) = \lim_{x\to\infty} \sum_{x^a<p_1\leqslant x^{a_1}} \frac{1}{p_1} \sum_{\substack{x^b<p_2\leqslant x^{b_1}\\ p_1p_2\leqslant x}} \frac{1}{p_2} = \lim_{x\to\infty} \sum_{x^a<p\leqslant x^{a_1}} \frac{S(p,b,b_1)}{p},$$

where $a_1 = a\mathrm{e}^{\alpha}$, $b_1 = b\mathrm{e}^{\beta}$ and $S(p,b,b_1)$ equals to $\beta$, $\ln(\ln x - \ln p) - \ln\ln x^b$ and 0 in the ranges $p < x^{1-b_1}$, $x^{1-b_1} \leqslant p < x^{1-b}$ and $p \geqslant x^{1-b}$ respectively. Using the standard formulas for sums over prime numbers we have for $0 < \tau \leqslant \lambda < 1$

$$\sum_{x^\tau<p\leqslant x^\lambda} \frac{1}{p} \ln\Big(1 - \frac{\ln p}{\ln x}\Big) = - \sum_{x^\tau<p\leqslant x^\lambda} \frac{1}{p} \sum_{k=1}^{\infty} \frac{1}{k}\Big(\frac{\ln p}{\ln x}\Big)^k$$

$$= \mathrm{Li}_2(\tau) - \mathrm{Li}_2(\lambda) + \mathrm{O}\Big(\frac{1}{\ln x}\Big) \tag{6}$$

Consider the function $\theta(a,b)$ separately on the subsets $W_i$ of the rectangle $W$:

$$W_1 = \{(a,b) \in W : a + b \geqslant 1\},$$
$$W_2 = \{(a,b) \in W : a + b \leqslant 1,\ a + b_1 \geqslant 1,\ a_1 + b \geqslant 1\},$$
$$W_3 = \{(a,b) \in W : a + b_1 \geqslant 1,\ a_1 + b \leqslant 1\},$$

$$W_4 = \{(a,b) \in W : a + b_1 \leqslant 1, a_1 + b \geqslant 1\},$$
$$W_5 = \{(a,b) \in W : a + b_1 \geqslant 1, a_1 + b \leqslant 1, a_1 + b_1 \geqslant 1\},$$
$$W_6 = \{(a,b) \in W : a_1 + b_1 \leqslant 1\}.$$

Having in mind (6), it is straightforward to evaluate the function $\theta(a,b)$. Thus $\theta(a,b) = h_i(a,b)$ for $(a,b) \in W_i$, $1 \leqslant i \leqslant 6$. The functions $h_i(a,b)$ are defined as follows

$$h_1(a,b) = 0, \quad h_2(a,b) = \ln b \ln \frac{a}{1-b} + \mathrm{Li}_2(a) - \mathrm{Li}_2(1-b),$$
$$h_3(a,b) = -\alpha \ln b + \mathrm{Li}_2(a) - \mathrm{Li}_2(a_1),$$
$$h_4(a,b) = (\beta + \ln b) \ln \frac{1-b_1}{a} + \ln b \ln \frac{a}{1-b} + \mathrm{Li}_2(1-b_1) - \mathrm{Li}_2(1-b),$$
$$h_5(a,b) = (\beta + \ln b) \ln \frac{1-b_1}{a} - \alpha \ln b + \mathrm{Li}_2(1-b_1) - \mathrm{Li}_2(a_1),$$
$$h_6(a,b) = \alpha\beta.$$

We note that $h_i$ is continuous on $W_i$ and $h_i(a,b) = h_j(a,b)$ when $(a,b) \in W_i \cap W_j$. Therefore the function $\theta(a,b)$ is continuous in the whole rectangle $W = W_1 \cup \ldots \cup W_6$ for each fixed pair $\alpha$, $\beta$. It is not difficult to show that

$$\theta_1 = \theta\Big(\frac{\mathrm{e}^{-\alpha}}{2}, \frac{2\mathrm{e}^{-\beta}}{3}\Big) \leqslant \theta \leqslant \theta\Big(\frac{1}{3}, \frac{1}{2}\Big) = \theta_2.$$

This completes the first part of the proof of Theorem 3.

II. Let $G$ be a distribution with characteristic function (5). Bearing in mind the continuity of $\theta(a,b)$ we have that for any real number $\theta \in [\theta_1, \theta_2]$ there exists a point $(a,b) \in W$ where $\theta(a,b) = \theta$. Now we are ready to construct the set of strongly additive functions $f_x$. Defining $A_x$ by

$$A_x = \big(x^a, x^{a\mathrm{e}^\alpha}\big] \cup \big(x^b, x^{b\mathrm{e}^\beta}\big] \cup \big(x^{2/3}, x^{2\mathrm{e}^\gamma/3}\big],$$

we take

$$f_x(p) = \varepsilon_1 \mathbf{1}_{\{P_1\}}(p) + \varepsilon_2 \mathbf{1}_{\{P_2\}}(p) + \mathbf{1}_{A_x}(p)$$

for any prime number $p$. It follows from Theorem 1 that $\nu_x\big(f_x(m) < u\big)$ converges to the distribution $G$. Hence $G \in \mathcal{M}$ and the proof of Theorem 3 is complete.

## References

[1] J. Šiaulys, On the distributions of additive functions, *LMD mokslo darbai*, **3**: supplement of *Liet. Matem. rink.*, Vilnius, MII, 104–109 (1999).
[2] J. Šiaulys, Factorial moments for distributions of additive functions (to appear in *Liet. Matem. Rink.*).

# Apie adityviųjų funkcijų skirstinių ribas

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Nagrinėjama stipriai adityviųjų funkcijų šeima $\{f_x, x \geqslant 2\}$, kuriai $f_x(p) \in \{0,1\}$ visiems pirminiams skaičiams $p$. Straipsnyje aprašomi tokių funkcijų skirstinių ribiniai dėsniai su nešėjais $\{0,1\}$ arba $\{0,1,2\}$.