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On integer parts of some sequences

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1. Introduction

Let a_n be some increasing sequence of positive real numbers. With this sequence we generate for each $\alpha > 0$ a sequence of natural numbers $A(n, \alpha)$. Examples are: $A(n, \alpha) = [a_n\alpha]$, $[a_n^\alpha]$, $[\alpha^{a_n}]$.

With some sequence of subsets of natural numbers $S_1, S_2, \ldots$ let us define

$$\mathcal{A}(\alpha) = \{n : A(n, \alpha) \in S_n\}.$$

We are interested on the conditions which imply that for almost all $\alpha > 0$ the sets $\mathcal{A}(\alpha)$ are infinite. An interesting instance of this problem was investigated by G. Harman [1]. With $S_1 = S_2 = \ldots = \mathcal{P}$ being the set of all prime numbers he proved that $\mathcal{A}(\alpha)$ are infinite for almost all $\alpha > 0$ if and only if the series

$$\sum_{n=1}^{\infty} \frac{1}{\log a_n} \Big(\sum_{\substack{m \leqslant n \\ |a_m - a_n| < 1}} 1 \Big)^{-1}$$

diverges.

We shall consider the outlined problem with

$$S_n = \{m : m \equiv m_n \pmod{M_n}\},$$

where $0 \leqslant m_n < M_n$ are given natural numbers. Note, that the case of bounded M_n was investigated in [3].

Theorem. *Let a_n be a sequence of positive numbers, $0 \leqslant m_n < M_n$, $M_n/a_n \to 0$ as $n \to \infty$. Let $A(n, \alpha) = [a_n\alpha]$ and*

$$\mathcal{A}(\alpha) = \big\{n : A(n, \alpha) \equiv m_n \pmod{M_n}\big\}.$$

If the series

$$\sum_{n=1}^{\infty} \frac{1}{M_n} \tag{1}$$

*Darbą remia Lietuvos mokslo ir studijų fondas.

converges, then $\mathcal{A}(\alpha)$ is finite for almost all $\alpha > 0$. If the series (1) *diverges and*

$$M_n \ll \frac{a_n}{a_m}, \quad m \leqslant n - \Delta_n, \quad \Delta_n = \sum_{m \leqslant n} \frac{1}{M_n}, \tag{2}$$

then $\mathcal{A}(\alpha)$ is infinite for almost all $\alpha > 0$.

For increasing sequence a_n the condition (2) may be replaced by

$$M_n \ll \frac{a_n}{a_{n-[\Delta_n]}}. \tag{3}$$

If $a_n = q^n$, then (3) is satisfied with $M_n \ln q \cdot n$.

2. Proof of the theorem

Our main tool is the following proposition. The Lebesque measure on the real line is denoted by λ.

Lemma ([2], Lemma 6.1, p.171). *Let J be a subinterval of the real line and $\mathcal{D}_n$ be a sequence of subsets of J. For each open interval $I \subset J$ suppose that there is a sequence of sets $\mathcal{B}_n \subset \mathcal{D}_n \cap I$ such that*

$$\sum_{n=1}^{\infty} \lambda(\mathcal{B}_n) = +\infty$$

and

$$\limsup_{N \to \infty} \Big(\sum_{n \leqslant N} \lambda(\mathcal{B}_n) \Big)^2 \Big(\sum_{m,n \leqslant N} \lambda(\mathcal{B}_n \cap \mathcal{B}_m) \Big)^{-1} \geqslant \delta \lambda(I), \tag{4}$$

where δ is a positive constant independent on I. Then almost all $\alpha \in J$ belong to infinitely many $\mathcal{D}_n$.

With the notation of our theorem we set

$$\mathcal{D}_n = \{\alpha : A(n, \alpha) \equiv m_n \pmod{M_n}\}.$$

The condition $\alpha \in \mathcal{D}_n$ is equivalent then to the existence of some natural number s satisfying

$$m_n + sM_n \leqslant a_n \alpha < m_n + 1 + sM_n, \quad \text{or}$$
$$\alpha \in \left[\frac{m_n}{a_n} + s\frac{M_n}{a_n}, \frac{m_n + 1}{a_n} + s\frac{M_n}{a_n} \right).$$

Then

$$\mathcal{D}_n = \bigcup_{s\geqslant 0} J(n,s), \quad J(n,s) = \left[\frac{m_n}{a_n} + s\frac{M_n}{a_n}, \frac{m_n+1}{a_n} + s\frac{M_n}{a_n}\right).$$

We fix now an interval $I = (a, a+b), a, b > 0$ and denote $\mathcal{B}_n = \mathcal{D}_n \cap I$. The set $\mathcal{B}_n$ is a union of some intervals $J(n,s)$, two of them are shortened, if necessary, by taking the lower (upper) range equal to a $(a+b)$.

The number of intervals in the union is equal to the number of integers s, satisfying $a \leqslant (m_n + sM_n)/a_n < a+b$ increased by 1. Hence

$$\begin{aligned}\lambda(\mathcal{B}_n) &= \frac{1}{a_n}\#\left\{s : a \leqslant s\frac{M_n}{a_n} + \frac{m_n}{a_n} < a+b\right\}\\ &\quad + \mathrm{O}\left(\frac{1}{a_n}\right) = \frac{b}{M_n} + \mathrm{O}\left(\frac{1}{a_n}\right) = \frac{b}{M_n}(1+\mathrm{o}(1)).\end{aligned}$$

It follows now from the Borel-Cantelli lemma that the convergence of the series (1) imply that $\lambda(\limsup \mathcal{B}_n) = 0$, which means that for almost all $\alpha \in I$ the sets $\mathcal{A}(\alpha)$ are finite. This proves the first part of Theorem.

For the second part we need to prove (4) with $\mathcal{B}_n$ defined as above. Let

$$L(N) = \sum_{1\leqslant k,l\leqslant N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) = \sum_{k\leqslant N} \lambda(\mathcal{B}_k) + \sum_{1\leqslant k<l\leqslant N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l). \tag{5}$$

For the first sum we have

$$\sum_{k\leqslant N} \lambda(\mathcal{B}_k) = \lambda(I)(1+o(1)) \sum_{k\leqslant N} \frac{1}{M_k}, \quad \lambda(I) = b. \tag{6}$$

To prove (4), we need an upper bound for $L(N)$. It follows from (5) and (6) that it suffices to prove that with some $C > 0$ independent of I

$$\sum_{1\leqslant k<l\leqslant N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) < C\lambda(I)\bigg(\sum_{m\leqslant N} \frac{1}{M_n}\bigg)^2 \tag{7}$$

holds. Consider now the summands on the left-hand side of (7). Let

$$\mathcal{B}_k = \cup_s{}^* J(k,s), \quad \mathcal{B}_l = \cup_t{}^* J(l,t),$$

where $*$ indicates that the sums are taken over some appropriate subsets of s and t and two intervals in each union are, if necessary, shortened.

We fix some s and consider when

$$J(k,s) \cap J(l,t) \neq \varnothing. \tag{8}$$

A sufficient condition for this is

$$\frac{m_k}{a_k} + s\frac{M_k}{a_k} \leqslant \frac{m_l}{a_l} + t\frac{M_l}{a_l} < \frac{m_k + 1}{a_k} + s\frac{M_k}{a_k}. \tag{9}$$

Both inequalities are also necessary for all t, except, probably for one case. The number of t satisfying (9) is bounded by $a_l/(a_k M_k) + 1$, hence, the number of non-empty intersections in (8) is

$$\frac{a_l}{a_k M_l} + \mathrm{O}(1).$$

As a consequence we obtain

$$\lambda(J(k,s) \cap \mathcal{B}_l) = \frac{1}{a_l}\left(\frac{a_l}{a_k M_l} + \mathrm{O}(1)\right).$$

How large is the number of intervals $\mathcal{B}_k$ consists of? For all these intervals except one the inequality

$$a < \frac{m_k}{a_k} + s\frac{M_k}{a_k} < a + b$$

must hold. The number of intervals is then bounded by $ba_k/M_k + \mathrm{O}(1)$. We have now

$$\begin{aligned}\lambda(\mathcal{B}_k \cap \mathcal{B}_l) &= \left(\frac{ba_k}{M_k} + \mathrm{O}(1)\right)\left(\frac{1}{a_k M_l} + \mathrm{O}\left(\frac{1}{a_l}\right)\right) \\ &= \frac{b}{M_k M_l}\left(1 + \mathrm{O}\left(\frac{M_k}{ba_k} + \frac{M_k M_l}{ba_l} + \frac{a_k M_l}{a_l}\right)\right).\end{aligned}$$

We use this result for $k_0 \leqslant k \leqslant l - \Delta_l$. Then $M_l \ll a_l/a_k$, and

$$\lambda(\mathcal{B}_k \cap \mathcal{B}_l) \ll \frac{b}{M_k M_l} = \frac{\lambda(I)}{M_k M_l} \tag{10}$$

with the constant in $\ll$ independent of I. For $k \leqslant l$ not in the range $k_0 \leqslant k < l - \Delta_l$ we use the trivial bound

$$\lambda(\mathcal{B}_k \cap \mathcal{B}_l) \ll \lambda(\mathcal{B}_l) \ll \frac{b}{M_l}. \tag{11}$$

We are now ready to prove (7). Using the bounds (10) and (11) we have

$$\begin{aligned}\sum_{1 \leqslant k < l \leqslant N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) &= \sum_{l=1}^{n} \sum_{k \in [k_0, l - \Delta_n]} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) + \sum_{l=1}^{n} \sum_{k \notin [k_0, l-\Delta_n]} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) \\ &\ll b \sum_{1 \leqslant k \leqslant l \leqslant N} \frac{1}{M_k M_l} + b\sum_{l=1}^{n}(k_0 + \Delta_l)\frac{1}{M_l} \ll b\left(\sum_{n \leqslant N} \frac{1}{M_n}\right)^2.\end{aligned}$$

The theorem is now proved.

References

[1] G. Harman, Metrical theorems on prime values of the integer parts of real sequences, *Proc. London Math. Soc. (3)*, **75**, 481–496 (1997).
[2] G. Harman, *Metric Number Theory*, Clarendon Press, Oxford (1998).
[3] V. Stakėnas, Tikimybinė skaičių teorija ir kontinumas, *Lietuvos matematikų draugijos mokslo darbai*, **3**, 93–99 (1999).

Sveikosios tam tikrų sekų dalys

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Darbe nagrinėjamos natūraliųjų skaičių sekos $[a_n\alpha]$, čia a_n – teigiamų realiųjų skaičių skaičių seka $\alpha > 0$. Įrodytoje teoremoje tvirtinama, kad esant tam tikroms sąlygoms beveik visiems $\alpha > 0$ skaičiai $[a_n\alpha]$ tenkina lyginius $[a_n\alpha] \equiv m_n \pmod{M_n}$ be galo daug kartų.