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Some inequalities in the arithmetic semigroups

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1. Introduction

By $\mathcal{N}, \mathcal{R}, \mathcal{C}$ we denote the sets of natural, real, and complex numbers, respectively and $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$.

Let $\mathcal{S}$ be an additive arithmetical semigroup, with degree mapping $\partial : \mathcal{S} \to \mathcal{N}_0$ (more about it see in [1], [3]), satisfying the following assumption about a total number $S(n)$ of elements of degree n in $\mathcal{S}$.

Axiom: *There exist the constants $A > 0, \gamma > 1$ and $q > 1$ (all depending on $\mathcal{S}$), such that*

$$S(n) = Aq^n + O\Big(\frac{q^n}{n^\gamma}\Big) \ \ as\ \ n \to \infty.$$

We shall construct a probability space, which enable to consider the arithmetical functions as sums of independent random variables.

Define the set $\mathcal{S}_n$, $n \in \mathcal{N}$ by

$$\mathcal{S}_n = \{a \in \mathcal{S}; \forall p | a \Rightarrow \partial(p) \leqslant n\}.$$

where p throughout the paper means a prime element of S.

By $\mathcal{A}$ we denote the class of all subsets of $\mathcal{S}$. Then for each $A \in \mathcal{A}$ we write

$$\nu_n(A) = \gamma_n \sum_{a \in A \cap \mathcal{S}_n} \frac{1}{q^{\partial(a)}}, \ \ \gamma_n := \prod_{\partial(p) \leqslant n} \left(1 - \frac{1}{q^{\partial(p)}}\right). \tag{1}$$

It is clear, that

$$\nu_n(\mathcal{S}_n) = 1.$$

Thus, the tripple $\{\mathcal{S}, \mathcal{A}, \nu_n\}$ is a probability space.

By $\overline{U}$ we denote the supplement of the set U. It follows from our Axiom, that $q = 1 + s,\ s > 0$. Let with some fixed u

$$Q_1 := \{p^k \in \mathcal{S};\ \partial(p) \leqslant n,\ \ k \geqslant 1,\ \ \exists u \in U,\ \ (u, p) = 1,\ \ p^k u \in U\}.$$

Lemma 1. *Suppose that $U \subset \mathcal{S}$ is such, that*

$$a := \nu_n(\overline{U}) < \frac{1}{48}\left(\frac{s}{s+1}\right)^2.$$

Then we have

$$\sum_{p^k \in Q_2} \frac{1}{q^{k\partial(p)}} \leqslant \frac{3(s+1)}{s} a,$$

where

$$Q_2 = \left\{p^k \in \mathcal{S};\ \partial(p) \leqslant n,\ k \geqslant 1,\ p^k \notin Q_1\right\}.$$

Proof of Lemma 1. Let $p_1^{k_1} \in Q_2$. We set

$$V_{p_1} = \left\{p_1^{k_1}u; u \in U, (p_1, u) = 1\right\}.$$

It is clear, that $V_{p_1} \cap U = \emptyset$. Then $V_{p_1} \subset \overline{U}$. It follows that

$$\nu_n(V_{p_1}) \leqslant a.$$

Further, for $p_1^{k_1} \in Q_2$ we have

$$\begin{aligned}\nu_n(V_{p_1}) &= \gamma_n \sum_{u \in U \cap \mathcal{S}_n} \frac{1}{q^{k_1\partial(p_1)+\partial(u)}} - \gamma_n \sum_{\substack{u \in U \cap \mathcal{S}_n, \\ u \equiv 0 (\mathrm{mod}\, p_1)}} \frac{1}{q^{\partial(u)+\partial(p_1)k_1}} \\ &\geqslant \frac{\nu_n(U)}{q^{k_1\partial(p_1)}} - \frac{1}{q^{(k_1+1)\partial(p_1)}}.\end{aligned}$$

From the last relation and equality

$$\nu_n(U) = 1 - \nu_n(\overline{U}) = 1 - a,$$

we obtain

$$\nu_n(V_{p_1}) \geqslant \frac{1}{q^{k_1\partial(p_1)}}\left\{1 - a - \frac{1}{q}\right\}. \qquad (2)$$

Thus, it follows

$$a \geqslant \frac{s}{q^{k_1\partial(p_1)}2(1+s)}.$$

The assumptions of the Lemma and the last inequality yield

$$\frac{1}{q^{k_1 \partial(p_1)}} \leqslant \frac{2(1+s)a}{s} \leqslant \frac{s}{24(1+s)}. \tag{3}$$

Let p_1, p_2 be two distinct prime elements. If $p_1 = p_2$ and $k_1 \neq k_2$, here $p_1^{k_1}, p_2^{k_2} \in Q_2$, then $V_{p_1} \cap V_{p_2} = \emptyset$. In the case $p_1 \neq p_2$ we have,

$$V_{p_1} \cap V_{p_2} \subset \left\{m \in \mathcal{S}; m \equiv 0(\operatorname{mod} r^{k_1})\right\},$$

here $r = p_1 p_2$. Therefore in both cases we have

$$\begin{aligned} \nu_n\left(V_{p1} \cap V_{p2}\right) &\leqslant \nu_n\left(m \in \mathcal{S}; m \equiv 0(\operatorname{mod} r^{k_1})\right) \leqslant \gamma_n \sum_{l \in \mathcal{S}_n} \frac{1}{q^{\partial(l)} q^{k_1 \partial(r)}} \\ &= \frac{1}{q^{k_1 \partial(r)}}. \end{aligned} \tag{4}$$

Let Q be an arbitrary subset of Q_2. Define

$$V = \bigcup_{p^k \in Q} V_p, \quad b := \sum_{p^k \in Q} \frac{1}{q^{k \partial(p)}}.$$

It follows from the estimations (2) and (4) that

$$\nu_n(V) \geqslant \sum_{p_1^{k_1} \in Q} \nu_n(V_{p_1}) - \sum_{p_1^{k_1}, p_2^{k_2} \in Q} \nu_n(V_{p_1} \cap V_{p_2}) \geqslant b\left(\frac{s}{2(1-s)} - b\right). \tag{5}$$

The equality $V \cap U = \emptyset$ yields that $\nu_n(V) \leqslant a$.

Suppose, that

$$b \leqslant \frac{s}{8(1+s)}.$$

Using this assumption in (5) we arrive at the relation

$$a \geqslant b\left(\frac{s}{2(1+s)} - \frac{s}{8(1+s)}\right) \geqslant \frac{sb}{3(1+s)}.$$

Choosing $Q = Q_2$, from the last estimation we conclude

$$\sum_{p^k \in Q_2} \frac{1}{q^{k \partial(p)}} \leqslant \frac{3(1+s)a}{s}.$$

We see, that the last relation yields validity of the Lemma 1.

Further, assume that

$$b > \frac{s}{8(1+s)}.$$

Let the set $Q^* \subset Q_2$ satisfy the following inequality:

$$b = b(Q^*) = \sum_{p^k \in Q^*} \frac{1}{q^{k\partial(p)}} \leqslant \frac{s}{8(1+s)} \tag{6}$$

and for all

$$p^k \in Q_2 \setminus Q^*,\ b + \frac{1}{q^{k\partial(p)}} > \frac{s}{8(1+s)}.$$

Applying the estimation (3) we have

$$b > \frac{s}{8(1+s)} - \frac{1}{q^{k\partial(p)}} \geqslant \frac{s}{12(1+s)}.$$

The last estimation together with (5) and (6) yield

$$a \geqslant \frac{1}{32}\left(\frac{s}{1+s}\right)^2.$$

Thus, we arrive at the contradiction, because

$$a \geqslant \left(\frac{s}{1+s}\right)^2 \frac{1}{32} > \left(\frac{s}{1+s}\right)^2 \frac{1}{48}.$$

Lemma 1 is proved.

For the set $U \subset \mathcal{S}$ we define the set $V = V(U)$ by

$$V = \{\ a \in \mathcal{S};\ au_3 = u_1u_2,\ u_2 = u_3k,\ (u_3,k) = 1,\ (u_1,k) = 1,\ u_i \in U, i = 1,2,3\}. \tag{7}$$

Lemma 2. *Suppose that $U \subset \mathcal{S}$ and the set V is defined in* (7) *. Then the equality*

$$\mu_n(\overline{V}) = O_s(\nu_n(\overline{U}))$$

holds. Here $s = q - 1,\ \mu_n(A) = (1/S(n)) \sum_{\substack{m \in A \cap S,\\ \partial(m) = n}} 1.$

Proof of Lemma 2. We begin proving the lemma under an auxiliary condition

$$\nu_n(\overline{U}) = a < \frac{1}{48}\left(\frac{s}{s+1}\right)^2.$$

Assume that $m = p^k l$, when $(p, l) = 1$, $p^k \in Q_1$, $l \in U$. Then for each $u \in U$ with conditions: $(p, u) = 1$ and $p^k u \in U$ we have

$$mu = lup^k.$$

Therefore $m \in V$. It is clear, that for each $m = p^k l$, where $(p, l) = 1$, $p^k \in Q_2$ or $l \in \overline{U}$ we have $m \in \overline{V}$. Further

$$\sum_{\substack{\partial(m)\leqslant n\\ m\in\overline{V}}} \partial(m) \leqslant \sum_{\substack{\partial(l)\leqslant n\\ l\in\overline{U}}} \sum_{\partial(p^k)\leqslant n-\partial(l)} \partial(p^k) + \sum_{\substack{p^k\in Q_2\\ \partial(p^k)\leqslant n}} \partial(p^k) \sum \partial(l) \leqslant n - \partial(p^k)1$$
$$= S_1 + S_2. \tag{8}$$

Applying Lemma 1 we obtain

$$\sum_{k\partial(p)\leqslant n-\partial(l)} \partial(p^k) \leqslant q^{n-\partial(l)}, \quad \sum_{\substack{m\in\overline{U},\\ \partial(m)\leqslant n}} \frac{1}{q^{\partial(m)}} \leqslant \gamma_n^{-1}\left(\gamma_n \sum_{m\in S_n\cap\overline{U}} \frac{1}{q^{\partial(m)}}\right) = O(na).$$

It follows from these two inequalities, that

$$S_1 = O\left(q^n an\right). \tag{9}$$

Let us consider the sum S_2. Making use of Lemma 8 [3] and Lemma 1 we assert

$$S_2 \leqslant \sum_{\substack{p^k\in Q_2,\\ \partial(p^k)\leqslant n}} \partial(p^k) q^{n-\partial(p^k)} = O_s\left(nq^n a\right).$$

This estimation together with (9) and (8) yield

$$\sum_{\substack{m\in\overline{V},\\ \partial(m)\leqslant n}} \partial(m) = O_s\left(q^n an\right).$$

Further, making use of the last estimation we obtain

$$\mu_n(\overline{V}) \leqslant \frac{1}{nS(n)} \sum_{\substack{m\in\overline{V}\cap S,\\ \partial(m)\leqslant n}} \partial(m) = O\left(a\right).$$

It follows

$$\mu_n(\overline{V}) = O_s\Big(\nu_n(\overline{U})\Big).$$

The case

$$\nu_n(\overline{U}) > \frac{1}{48}\left(\frac{s}{s+1}\right)^2$$

is obvious. Lemma 2 is proved.

We define

$$\xi_p = \xi_p(m) = g_n(p^k); \text{ if } p^k||m, k \geqslant 0, \ m \in \mathcal{S}, n \in \mathcal{N}. \tag{10}$$

Here $g_n(m)$ is some sequence of real-valued arithmetic functions. The reader can easy prove the following

Theorem 1. *The functions* $\xi_p; \partial(p) \leqslant n$, *defined in* (10), *are independent random variables in the space* $\{\mathcal{S}, \mathcal{A}, \nu_n\}$, *distributed by*

$$\nu_n\big(\xi_p = g_n(p^k)\big) = \frac{1}{q^{k\partial(p)}}\Big(1 - q^{-\partial(p)}\Big), \tag{11}$$

$k \geqslant 0,\ \partial(p) \leqslant n, n \in \mathcal{N}$.

We set

$$G_n(m,u) = \prod_{\partial(p)\leqslant z(u), p^\alpha||m} g_n(p^\alpha), \quad m \in \mathcal{S};$$

$$Z_n(u) = \prod_{\partial(p)\leqslant z(u)} |\xi_p|^{\frac{1}{\beta(n)}} \operatorname{sgn}\xi_p e^{-\alpha(u)},$$

where the function $z(u), u \in [0,1]$ is some real-valued monotonic function with condition $z(1) = n,\ \beta(n) \to \infty$ as $n \to \infty$, and $\xi(p),\ \partial(p) \leqslant n$ are random variables defined by (11).

Theorem 2. *For each* $0.5 < \epsilon < 1$ *we have*

$$\mu_n\Big(\sup_{0\leqslant u\leqslant 1} |G_n(m,u) - 1| \geqslant \epsilon\Big) = O_s\Big(\nu_n\big(\sup_{0\leqslant u\leqslant 1} |Z_n(u) - 1| \geqslant \frac{\epsilon}{8}\big)\Big).$$

Proof of Theorem 2. Set

$$U = \big\{m \in \mathcal{S};\ \sup_{0\leqslant u\leqslant 1} |G_n(m,u) - 1| < \frac{\epsilon}{8}\big\},$$

here $u \in [0, 1]$ is a fixed number. Let the set V be defined as in (7). It means, that for each $m \in V$ there exist the numbers $n_1, n_2, n_3 \in U$ such, that $mn_3 = n_1 n_2$, where $n_3 || n_2, n_2 = n_3 l, (n_1, l) = 1$. Since each number of $\mathcal{S}$ has unique decomposition into the power of prime numbers, then multiplicity of $G_n(\circ, u)$ implies relation

$$G_n(m, u) = G_n(n_1, u) G_n(l, u) = G_1 G_2 G_3^{-1},$$

where $G_i = G_n(n_i, u); n_i \in U, i = 1, 2, 3$.

Use of the identity

$$1 - \prod_{i=1}^{k} b_i = \sum_{j=1}^{k} (1 - b_j) \prod_{l=j+1}^{k} b_l$$

leads to the inequalities

$$\begin{aligned} |G_n(m, u) - 1| &\leqslant |1 - G_1||G_2 G_3^{-1}| + |1 - G_2||G_1 G_3^{-1}| + |1 - G_2||G_2 G_1 G_3^{-1}| \\ &\leqslant (1 + \epsilon)^2)(|1 - G_1| + |1 - G_2| + 2|1 - G_3|) \leqslant \epsilon. \end{aligned}$$

Therefore $|G_n(m, u) - 1| \leqslant \epsilon$, $\forall m \in U$. It yields

$$V \subset \{ m \in \mathcal{S};\ \sup_{0 \leqslant u \leqslant 1} |G_n(m, u) - 1| \leqslant \epsilon \}.$$

The last relation together with Lemma 2 complete the proof of the Theorem 2.

References

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Keletas nelygybių aritmetinėse pusgrupėse

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Šiame darbe pateiktas Ružos nelygybės analogo aritmetinėse pusgrupėse įrodymas. Be to, pateiktas pavyzdys, kaip galima pritaikyti šią nelygybę aritmetinių bei atsitiktinių multiplikacinių procesų tikimybių palyginimui.