*LMD mokslo darbai*, III tomas, 68–74


# Odd logarithmic moments of the Riemann zeta-function

Antanas LAURINČIKAS (VU, ŠU)*
*e-mail: antanas.laurincikas@maf.vu.lt*

## 1. Introduction

Let $\zeta(s)$, $s = \sigma + it$, as usual, denote the Riemann zeta-function. In the investigation of the value-distribution of $\zeta(s)$ an important role is played by its logarithmic moments

$$J_k(\sigma, T) \overset{def}{=} \int_0^T \big(\log|\zeta(\sigma+it)|\big)^k \,\mathrm{d}t$$

for any integer $k$. Here $\sigma = 1/2$ or $\sigma = \sigma_T$ and tends to 1/2 as $T \to \infty$. In this note we study the moments $J_{2k-1}(1/2, T)$ for any positive integer $k$. Denote by $B_\eta$ a number bounded by a constant depending on $\eta$.

**Theorem 1.** *For $T \to \infty$ we have*

$$J_{2k-1}(1/2, T) = B_k T (\log\log T)^{k-1}.$$

Note that [1], for $T \to \infty$,

$$J_{2k}(1/2, T) \sim a(k) T (\log\log T)^k,$$

where $a(k)$ is an explicitly given function.

## 2. Auxiliary results

Let $\Lambda(m)$ stand for the von Mangoldt function, and let, for $x \geqslant 2$, $\Lambda_x(m)$ denote the Sellberg [3] function, i.e.,

$$\Lambda_x(m) = \begin{cases} \Lambda(m), & 1 \leqslant m \leqslant x, \\ \Lambda(m)\dfrac{\log^2\frac{x^3}{m} - 2\log^2\frac{x^2}{m}}{2\log^2 x}, & x \leqslant m \leqslant x^2, \\ \Lambda(m)\dfrac{\log^2\frac{x^3}{m}}{2\log^2 x}, & x^2 \leqslant m \leqslant x^3. \end{cases}$$

Moreover, for $t > 0$, let

$$\sigma_{x,t} = \frac{1}{2} + 2\max\left(\max_{\varrho}\left|\beta - \frac{1}{2}\right|, \frac{1}{\log x}\right),$$

where $\varrho$ runs over all zeros $\beta + i\gamma$ of $\zeta(s)$ for which

$$|t-\gamma| \leqslant \frac{x^{3|\beta-1/2|}}{\log x}.$$

Now we can define the functions $E_j(t)$, $j = 1, \ldots, 5$, by

$$E_1(t) = \sum_{p<x^3} \frac{\Lambda(p) - \Lambda_x(p)}{\sqrt{p}\log p} p^{-it},$$

$$E_2(t) = \sum_{p<x^{3/2}} \frac{\Lambda_x(p^2)}{p\log p} p^{-2it},$$

$$E_3(t) = \left(\sigma_{x,t} - \frac{1}{2}\right)x^{\sigma_{x,t}-1/2}\int_{1/2}^{\infty} x^{1/2-\sigma}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|\mathrm{d}\sigma,$$

$$E_4(t) = \left(\sigma_{x,t} - \frac{1}{2}\right)\log U,$$

$$E_5(t) = \sum_{|t-\gamma|\leqslant 1} \log\left|\frac{\sigma_{x,t} + it - \varrho}{1/2 + it - \varrho}\right|.$$

Suppose that $U \leqslant T$.

**Lemma 1.** *Let* $x = U^{1/(120k)}$, $k \in \mathbb{N}$. *Then*

$$\int_{U/2}^{U} |E_j(t)|^{2k}\mathrm{d}t = B_k U, \qquad j = 1, \ldots, 5.$$

*Proof.* For $j = 1, \ldots, 4$, the estimate of the lemma is given in [1], see also [2]. The case $j = 5$ can be found in [1], and, for $k = 1$, in [2].

**Lemma 2.** *Let* $x = U^{1/(120k)}$, $k \in \mathbb{N}$, *and* $t \in [U/2, U]$. *Then*

$$\log\left|\zeta\left(\frac{1}{2} + it\right)\right| = \sum_{p<x^3} \frac{\cos(t\log p)}{\sqrt{p}} + B\sum_{j=1}^{5}|E_j(t)|.$$

Proof of the lemma is given in [2].

**Lemma 3.** *Suppose that* $U^{1/(120k)} \leqslant y \leqslant U^{1/k}$, $k \in \mathbb{N}$. *Then*

$$\int_{U/2}^{U} \left| \log \left| \zeta\left(\frac{1}{2} + it\right) \right| - \sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}} \right|^{2k-1} \mathrm{d}t = B_k U.$$

*Proof.* By Lemma 2 the integrand of the lemma does not exceed

$$B_k \left( \sum_{j=1}^{5} |E_j(t)|^{2k-1} + \left| \sum_{x^3 \leqslant p < y} \frac{1}{p^{1/2+it}} \right|^{2k-1} \right). \tag{1}$$

The conditions of the lemma, for $x^3 \leqslant p < y$, imply the estimate

$$1 = \frac{\log p}{\log y} \cdot \frac{\log y}{\log p} = B \frac{\log p}{\log y}.$$

Therefore, by Theorem 2.7.4 of [2] we obtain that

$$\int_{0}^{U} \left| \sum_{x^3 \leqslant p < y} \frac{1}{p^{1/2+it}} \right|^{2k-1} \mathrm{d}t = B\sqrt{U} \left( \int_{0}^{U} \left| \sum_{x^3 \leqslant p < y} \frac{1}{p^{1/2+it}} \right|^{4k-2} \mathrm{d}t \right)^{1/2} = B_k U.$$

Hence, from (1) and Lemma 1, applying the Cauchy inequality, we obtain the estimate of the lemma.

## 3. Proof of Theorem

We begin the proof with the integral

$$\int_{U/2}^{U} \left( \log \left| \zeta\left(\frac{1}{2} + it\right) \right| \right)^{2k-1} \mathrm{d}t.$$

We take $y = U^{1/(5k)}$ and put

$$\Delta_y(t) = \Delta(t) = \log \left| \zeta\left(\frac{1}{2} + it\right) \right| - \sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}}.$$

Then, clearly,

$$\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right|\right)^{2k-1} = \left(\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right)^{2k-1}$$
$$+\sum_{m=1}^{2k-1} C_{2k-1}^{m}\Delta^{m}(t)\left(\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right)^{2k-1-m}. \tag{2}$$

It is not difficult to see that the second term in (2) does not exceed

$$|\Delta(t)|\sum_{m=1}^{2k-1} C_{2k-1}^{m}|\Delta(t)|^{m-1}\left|\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right|^{2k-1-m}$$
$$= |\Delta(t)|\sum_{m=0}^{2k-2} C_{2k-1}^{m+1}|\Delta(t)|^{m}\left|\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right|^{2k-m-2}$$
$$= B_k|\Delta(t)|\left(\Delta(t)+\left|\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right|\right)^{2k-2}.$$

This, (2), Lemma 3 and the Hölder inequality yield

$$\int_{U/2}^{U}\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right|\right)^{2k-1}\mathrm{d}t - \int_{U/2}^{U}\left(\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right)^{2k-1}\mathrm{d}t$$
$$= B_k\int_0^U|\Delta(t)|^{2k-1}\mathrm{d}t + B_k\int_0^U|\Delta(t)|\left|\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right|^{2k-2}\mathrm{d}t$$
$$= B_k\int_0^U|\Delta(t)|^{2k-1}\mathrm{d}t + B_k\left(\int_0^U|\Delta(t)|^{2k-1}\mathrm{d}t\right)^{1/(2k-1)}$$
$$\times\left(\int_0^U\left|\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right|^{2k-1}\mathrm{d}t\right)^{1-1/(2k-1)}$$
$$= B_kU + B_kU^{1/(2k-1)}\left(\int_0^U\left|\sum_{p<y}\frac{\cos(t\log p)}{\sqrt{p}}\right|^{2k-1}\mathrm{d}t\right)^{1-1/(2k-1)}. \tag{3}$$

Now let

$$\eta = \eta(t) = \sum_{p<y}\frac{1}{p^{1/2+it}}.$$

Then we have

$$\sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}} = \frac{1}{2}(\eta + \bar{\eta}).$$

Consequently,

$$\int_{U/2}^{U} \left( \sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}} \right)^{2k-1} \mathrm{d}t = 2^{1-2k} \sum_{m=0}^{2k-1} C_{2k-1}^{m} \int_{U/2}^{U} \eta^m \bar{\eta}^{2k-1-m} \mathrm{d}t. \tag{4}$$

Obviously,

$$\begin{aligned}\int_{U/2}^{U} \eta^m \eta^{-2k-1-m} \mathrm{d}t &= \sum_{p_1,\dots,p_m<y} \sum_{q_1,\dots,q_{2k-1-m}<y} (p_1 \cdots p_m q_1 \cdots q_{2k-1-m})^{-1/2} \\ &\quad \times \int_{U/2}^{U} \left( \frac{p_1 \cdots p_m}{q_1 \cdots q_{2k-1-m}} \right)^{it} \mathrm{d}t \\ &= B \sum_{p_1,\dots,p_m<y} \sum_{q_1,\dots,q_{2k-1-m}<y} (p_1 \cdots p_m q_1 \cdots q_{2k-1-m})^{-1/2} \\ &\quad \times \left| \log \frac{p_1 \cdots p_m}{q_1 \cdots q_{2k-1-m}} \right|^{-1}. \end{aligned} \tag{5}$$

Since, for $m, n \in \mathbb{N}$,

$$\left| \log \frac{m}{n} \right| > \min\left( \frac{1}{m}, \frac{1}{n} \right),$$

we obtain that the logarithm in (5) is estimated as $By^{2k-1}$. Thus, in view of (4) and (5)

$$\begin{aligned}\int_{U/2}^{U} \left( \sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}} \right)^{2k-1} \mathrm{d}t &= By^{2k-1} \left( \sum_{p<y} \frac{1}{\sqrt{p}} \right)^{2k-1} \sum_{m=0}^{2k-1} C_{2k-1}^{m} \\ &= B_k y^{3k} = B_k U. \end{aligned} \tag{6}$$

It remains to estimate the integral

$$\int_{0}^{U} \left| \sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}} \right|^{2k-1} \mathrm{d}t \leqslant \sqrt{U} \left( \int_{0}^{U} \left| \sum_{p<y} \frac{\cos(t \log p)}{\sqrt{p}} \right|^{4k-2} \mathrm{d}t \right)^{1/2}. \tag{7}$$

Using the same notation as above, we find

$$\int_0^U \left|\sum_{p<y} \frac{\cos(t\log p)}{\sqrt{p}}\right)^{4k-2} \mathrm{d}t = 2^{2-4k} \sum_{m=0}^{4k-2} C_{4k-2}^m \int_0^U \eta^m \bar{\eta}^{4k-2-m} \mathrm{d}t$$

$$= B_k \int_0^U |\eta(t)|^{4k-2} \mathrm{d}t + B_k \sum_{\substack{m=0 \\ m\neq 2k-1}} C_{4k-2}^m \left| \int_0^U \eta^m \bar{\eta}^{4k-2-m} \mathrm{d}t \right|. \tag{8}$$

Similarly to (6) we obtain that the second term in (8) does not exceed $B_k U$. For the first term we have

$$\int_0^U |\eta(t)|^{4k-2} \mathrm{d}t = B_k U \sum_{\substack{p_1,\dots,p_{2k-1}<y \\ q_1,\dots,q_{2k-1}<y \\ p_1\cdots p_{2k-1}=q_1\cdots q_{2k-1}}} (p_1 \dots p_{2k-1})^{-1}$$

$$+B \sum_{\substack{p_1,\dots,p_{2k-1}<y \\ q_1,\dots,q_{2k-1}<y \\ p_1\cdots p_{2k-1}\neq q_1\cdots q_{2k-1}}} (p_1 \dots p_{2k-1})^{-1} \left| \log \frac{p_1 \dots p_{2k-1}}{q_1 \dots q_{2k-1}} \right|^{-1}$$

$$= B_k U \sum_{\substack{p_1,\dots,p_{2k-1}<y \\ q_1,\dots,q_{2k-1}<y \\ p_1\cdots p_{2k-1}=q_1\cdots q_{2k-1}}} (p_1 \dots p_{2k-1})^{-1} + B_k U. \tag{9}$$

By Lemma 2.7.3 from [2] the first term in (9) is

$$B_k U \left( \sum_{p<y} \frac{1}{p} \right)^{2k-1} + B_k U \left( \sum_{p<y} \frac{1}{p} \right)^{2k-3} = B_k U (\log\log T)^{2k-1}.$$

Thus, by (7), (8) and (9)

$$\int_0^U \left| \sum_{p<y} \frac{\cos(t\log p)}{\sqrt{p}} \right|^{2k-1} \mathrm{d}t = B_k U (\log\log T)^{(2k-1)/2}.$$

Now hence and from (3), (6) we have

$$\int_{U/2}^U \left( \log \left( \zeta\left(\frac{1}{2} + it\right) \right| \right)^{2k-1} \mathrm{d}t = B_k U (\log\log T)^{k-1}.$$

Taking $U = T/2^l$ and summing the last equality over $l = 0, 1, \dots$, we obtain the theorem.

## References

[1] D. Joyner, *Distribution Theorems for L-functions*, Longman Scientific and Technical (1986).
[2] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht (1996).
[3] A. Selberg, Contributions to the theory of the Riemann zeta-function, *Arch. Math. Naturvid.*, **48**, 89–155 (1946).

# Rymano dzeta funkcijos nelyginiai momentai

A. Laurinčikas

Straipsnyje gautas nelyginių logaritminių Rymano dzeta funkcijos momentų kritinėje tiesėje įvertis.