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On differential geometry of tangent and cotangent bundles of Riemannian manifold

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A large number of mathematicians investigate differential geometry of tangent and cotangent bundles of Riemannian manifold V_n. In works of japanesse investigators Yano, Sasaki and others [1, 2, 3] we may find excellent results on prolongations or lifts of some tensor fields and connections of V_n in these bundles.

In this paper we shall examine the tangent bundle $T(V_n)$ and cotangent bundle $T^c(V_n)$ of $n-$ dimensional Riemannian manifold V_n with metric g, which allows us to define the diffeomorphism $f : T(V_n) \to T^c(V_n)$. The diffeomorphism f is used to transfer tensor fields of $T(V_n)$ to ones in $T^c(V_n)$ and converse.

On the other hand, there exist lifts or continuations of functions, tensor fields of V_n to $T(V_n)$ and $T^c(V_n)$. We shall find some relations between these lifts in $T(V_n)$ and $T^c(V_n)$.

1. Let us have the $n-$ dimensional Riemannian manifold V_n with local system of coordinates x^i, $i, j, ... = 1, 2, ..., n$, and metric g_{ij}. Denote by $\nu_x(\nu^i) \in T_x V_n$ and $\widetilde{\nu}_x(\widetilde{\nu}_i) \in T^c_x V_n$ a vector and a covector, respectively, at a point $x(x^i) \in V_n$. Then a point X of the tangent bundle

$$T(V_n) = \bigcup_{x \in V_n} T_x V_n$$

has the coordinates $(X^A) = (x^i, \nu^i)$, and a point Y of the cotangent bundle

$$T^c(V_n) = \bigcup_{x \in V_n} T^c_x V_n$$

has the coordinates $(Y^A) = (x^i, \widetilde{\nu}_i)$, where $A, B, ... = 1, 2, ..., n, n+1, ..., 2n$.

Now we can define the diffeomorphism $f : T(V_n) \to T^c(V_n)$ by equations

$$\begin{aligned} &Y^i = X^i (= x^i), \\ &Y^{n+i} = g_{ij} X^{n+j} (\widetilde{\nu}_i = g_{ij} \nu^j). \end{aligned} \tag{1}$$

Let us have in $T(V_n)$ the curve $k : X^A = X^A(t)$. Then in $T^c(V_n)$ we have a corresponding curve $f \circ k$:

$$\begin{aligned} &Y^i = X^i(t), \\ &Y^{n+i} = g_{ij}(t) X^{n+j}(t). \end{aligned}$$

The tangent vector $V^A = \frac{dX^A}{dt}$ to the curve k and the tangent vector $\widetilde{V}^A = \frac{dY^A}{dt}$ to the corresponding curve $f \circ k$ in view of (1) are connected by the formulas

$$\widetilde{V}^i = V^i,$$
$$\widetilde{V}^{n+i} = \partial_j g_{ik} V^j X^{n+k} + g_{ik} V^{n+k}. \tag{2}$$

A mapping $f^{-1} : T^c(V_n) \to T(V_n)$, defined by formulas

$$X^i = Y^i (= x^i),$$
$$X^{n+i} = \sum_j g^{ij} Y^{n+j} (\nu^i = g^{ij} \widetilde{\nu}_j),$$

transfers every vector $\widetilde{V}^A$ to the vector V^A, where

$$V^i = \widetilde{V}^i,$$
$$V^{n+i} = \sum_l g^{il} \widetilde{V}^{n+l} + \partial_j g^{il} g_{ik} \widetilde{V}^j X^{n+k}. \tag{3}$$

Let us have the vector $u^i(x^j)$ in a basic manifold V_n. Then in $T(V_n)$ there are some vector fields:

1) a complete lift $U_1(u^i, X^{n+j} ds \frac{\partial u^i}{\partial x^j})$;
2) a vertical lift $U_2(0, u^i)$;
3) a horizontal lift $U_3(u^i, -\Gamma^i_{jk} u^j X^{n+k})$,

where Γ^i_{jk} are Christoffel's symbols defined by the metric g_{ij}.

We shall find vectors $\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3$ in $T^c(V_n)$ corresponding to the vectors U_1, U_2, U_3 with respect to diffeomorphism f.

From (2) we can easily find the vector $\widetilde{U}_2(0, g_{ij} u^j)$. Analogously, $\widetilde{U}^i_1 = u^i$,

$$\widetilde{U}^{n+i}_1 = \partial_j g_{ik} u^j X^{n+k} + g_{ik} X^{n+j} \frac{\partial u^k}{\partial x^j} = \partial_j g_{ik} u^i \sum_l g^{kl} Y^{n+l}$$
$$+ g_{ik} \frac{\partial u^k}{\partial x^j} \sum_l g^{jl} Y^{n+l} = \sum_l g_{ik} Y^{n+l} (-\partial_j g^{kl} u^j + g^{lj} \frac{\partial u^k}{\partial x^j}),$$
$$\widetilde{U}^i_3 = u^i, \quad \widetilde{U}^{n+i}_3 = \partial_j g_{ik} u^j X^{n+k} - g_{ik} \Gamma^k_{jl} X^{n+j} u^l$$
$$= \sum_l g^{lk} Y^{n+l} u^j (\partial_j g_{ik} - g_{im} \Gamma^m_{jk}) = \sum_k \Gamma^k_{ji} Y^{n+k} u^i.$$

PROPOSITION 1. The vectors $\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3$ in $T^c(V_n)$, corresponding with respect to diffeomorphism f to the complete, vertical, horizontal lifts U_1, U_2, U_3 in $T(V_n)$ of the vector field $u^i(x^i)$

in the basic manifold V_n, are the following

$$\widetilde{U}_1(u^i, \sum_l g_{ik}Y^{n+l}(-\partial_j g^{kl}u^j + g^{lj}\frac{\partial u^k}{\partial x^j})), \quad \widetilde{U}_2(0, g_{ij}u^j),$$
$$\widetilde{U}_3(u^i, \sum_k \Gamma^k_{ji}Y^{n+k}u^j),$$

respectively.

2. Let us examine a complete lift G_{AB} of metric g_{ij} from manifold V_n to $T(V_n)$ [4]:

$$(G_{AB}) = \begin{pmatrix} X^{n+k}\frac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}. \tag{4}$$

Wel search such a metric $\widetilde{G}_{AB}$ in $T^c(V_n)$ for which the equality $\widetilde{G}_{AB}\widetilde{U}^A\widetilde{V}^B = G_{AB}U^AV^B, \forall U, V \in T(V_n)$ holds. From (2-4)

$$\begin{aligned}
&G_{AB}U^AV^B = G_{ij}U^iV^j + G_{i,n+j}U^iV^{n+j} + G_{n+i,j}U^{n+i}V^j \\
&+G_{n+i,n+j}U^{n+i}V^{n+j} = X^{n+k}\partial_k g_{ij}U^iV^j + g_{ij}\left(U^iV^{n+j} + U^iV^{n+j}\right) \\
&= X^{n+k}\partial_k g_{ij}\widetilde{U}^i\widetilde{V}^j + g_{ij}\widetilde{U}^i\left(\partial_k g^{lj}\widetilde{V}^k g_{lm}X^{n+m} + \sum_l g^{lj}\widetilde{V}^{n+l}\right) \\
&+g_{ij}\widetilde{V}^j\left(\partial_k g^{li}\widetilde{U}^k g_{lm}X^{n+m} + \sum_l g^{lj}\widetilde{U}^{n+l}\right) \\
&= \widetilde{U}^i\widetilde{V}^j\left(X^{n+k}\partial_k g_{ij} + g_{ib}\partial_j g^{bl}g_{lm}X^{n+m} + g_{ja}\partial_i g^{al}g_{lm}X^{n+m}\right) \\
&+\widetilde{U}^i\widetilde{V}^{n+i} + \widetilde{U}^{n+j}\widetilde{V}^j = \widetilde{U}^i\widetilde{V}^jX^{n+m}\left(\partial_m g_{ij} - \partial_j g_{im} - \partial_i g_{jm}\right) + \delta_{ij}\widetilde{U}^i\widetilde{V}^{n+j} \\
&+\delta_{ij}\widetilde{U}^{n+i}\widetilde{V}^j = -2\sum_l Y^{n+l}\Gamma^l_{ji}\widetilde{U}^i\widetilde{V}^j + \delta_{ij}\widetilde{U}^i\widetilde{V}^{n+j} + \delta_{ij}\widetilde{U}^{n+i}\widetilde{V}^j.
\end{aligned}$$

Thus, matrix of the metric $\widetilde{G}_{AB}$ is

$$(\widetilde{G}_{AB}) = \begin{pmatrix} -2\sum_k Y^{n+k}\Gamma^k_{ij} & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}. \tag{5}$$

Proposition 2. The complete lift G_{AB} of the metric g_{ij} to the manifold $T(V_n)$ the diffeomorphism (1) transfers into the continuation of the metric g_{ij} to $T^c(V_n)$ (5) in the Yano-Patterson sence [3].

3. Let us consider the Sasaki metric [1] in $T(V_n)$ with a quadratic form $d\sigma^2 = ds^2 + g_{ij}D\nu^iD\nu^j$, where $ds^2 = g_{ij}dx^idx^j$, $D\nu^j = d\nu^j + \Gamma^j_{ik}\nu^idx^k$ is the covariant differential of

the vektor ν^j. Wel find the components of the Sasaki metric ${}^sG_{AB}$. We have

$$
\begin{aligned}
d\sigma^2 = {}^sG_{AB}dX^AdX^B &= g_{ij}dX^idX^j + g_{ij}(dX^{n+i} + \Gamma^i_{km}dX^k\nu^m) \\
&\times (dX^{n+j} + \Gamma^j_{lr}dX^l\nu^r) = g_{ij}dX^idX^j + g_{ij}dX^{n+i}dX^{n+j} \\
&+g_{ij}\Gamma^j_{km}\Gamma^j_{lr}\nu^m\nu^r dX^kdX^l + g_{ij}\Gamma^i_{km}\nu^m dX^k dX^{n+j} + g_{ij}dX^{n+i}dX^l\Gamma^j_{lr}\nu^r \\
= &\ (g_{ij} + g_{kl}\Gamma^k_{im}\Gamma^l_{jr}\nu^m\nu^r)dX^idX^j + g_{kj}\Gamma^k_{rm}\nu^m dX^idX^{n+j} \\
&+ g_{il}\Gamma^l_{jr}\nu^r dX^{n+i}dX^j + g_{ij}dX^{n+i}dX^{n+j}.
\end{aligned}
$$

Thus,

$$
({}^sG_{AB}) = \begin{pmatrix} g_{ij} + g_{kl}\Gamma^k_{im}\Gamma^l_{jr}\nu^m\nu^r & \Gamma^k_{il}g_{kj}\nu^l \\ \Gamma^k_{jl}g_{ik}\nu^l & g_{ij} \end{pmatrix}. \tag{6}
$$

We search for a metric ${}^s\widetilde{G}_{AB}$ in $T^c(V_n)$ such that ${}^s\widetilde{G}_{AB}dY^AdY^B = {}^sG_{AB}dX^AdX^B$. From (6), (1) we have

$$
\begin{aligned}
&{}^sG_{AB}dX^AdX^B = g_{ij}dX^idX^j + g_{ij}DX^{n+i}DX^{n+j} \\
&= g_{ij}dX^idX^j + g_{ij}(dX^{n+i} + \Gamma^i_{km}dx^kX^{n+m})(dX^{n+j} + \Gamma^j_{lr}dx^lX^{n+r}).
\end{aligned}
$$

Moreover, in virtue of (1)

$$
\begin{aligned}
D\nu^i &= DX^{n+i} = dX^{n+i} + \Gamma^i_{km}dx^kX^{n+m} = d(g^{ia}Y^{n+a}) + \Gamma^i_{km}dx^kg^{mb}Y^{n+b} \\
&= \partial_k g^{ia}dX^kY^{n+a} + g^{ia}dY^{n+a} + \Gamma^i_{km}dx^kg^{mb}Y^{n+b} \\
&= g^{ia}(dY^{n+a} - \Gamma^b_{ka}dx^kY^{n+b}) + dX^kY^{n+k}(\partial_k g^{ia} + \Gamma^i_{km}g^{ma} + \Gamma^a_{kb}g^{ib}) \\
&= g^{ia}DY^{n+a} + dX^kY^{n+a}\nabla_k g^{ia} = g^{ia}DY^{n+a}.
\end{aligned}
$$

Similarly $dX^{n+j} + \Gamma^j_{lr}dx^lX^{n+r} = g^{jb}DY^{n+b}$. Hence

$$
\begin{aligned}
&{}^sG_{AB}dX^AdX^B = g_{ij}dX^idX^j + g_{ij}g^{ia}DY^{n+a}g^{jb}DY^{n+b} \\
&= g_{ij}dX^idX^j + g^{ab}DY^{n+a}DY^{n+b} \\
&g_{ij}dY^idY^j + g^{ab}(dY^{n+a} - \Gamma^m_{ka}dX^kY^{n+m})(dY^{n+b} - \Gamma^r_{bl}dX^lY^{n+r}) \\
&= g_{ij}dY^idY^j + g^{ab}\Gamma^m_{ia}\Gamma^r_{bj}Y^{n+m}Y^{n+r}dY^idY^j - g^{aj}\Gamma^m_{ia}Y^{n+m}dY^idY^{n+j} \\
&-g^{ib}\Gamma^r_{bj}Y^{n+r}dY^{n+i}dY^j + g^{ab}dY^{n+a}dY^{n+b}.
\end{aligned}
$$

This shows that a matrix of the metric ${}^s\widetilde{G}_{AB}$ is

$$
({}^s\widetilde{G}_{AB}) = \begin{pmatrix} g_{ij} + g^{ab}\Gamma^m_{ia}\Gamma^r_{bj}Y^{n+i}Y^{n+j} & -g^{aj}\Gamma^m_{ia}Y^{n+m} \\ -g^{bj}\Gamma^m_{bj}Y^{n+m} & g^{ij} \end{pmatrix}. \tag{7}
$$

PROPOSITION 3. The diffeomorphism (1) maps the Sasaki metric (6) in $T(V_n)$ into the fundamental metric ${}^s\widetilde{G}_{AB}$ (7) in $T^c(V_n)$.

4. Finally we consider an 1-form $p = Y^{n+j}dY^j$ in $T^c(V_n)$, which is called the fundamental form [2]. The mapping $f^{-1} : T^c(V_n) \to T(V_n)$ transfers p into the 1-form $q = g_{ij}X^{n+i}dX^j$. Exterior differential $dp = dY^{n+j} \wedge Dy^j = \frac{1}{2}E_{AB}dY^A \wedge dY^B$ defines the skew - symmetric tensor E_{AB} with a matrix $(E_{AB}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where (I) is the unit $n-$ dimensional matrix. We shall find the corresponding exterior 2-form E' and the skew-symmetric tensor E'_{AB} in $T(V_n)$. In view of (1)

$$\begin{aligned} dY^{n+j} \wedge dY^j &= d(g_{jm}X^{n+m}) \wedge dX^j = (\partial_k g_{jm} dX^k X^{n+m} + g_{jm} dX^{n+m}) \wedge dX^j \\ &= \frac{1}{2}\partial_{[k}g_{j]m}X^{n+m}dX^k \wedge dX^j + \frac{1}{2}g_{jm}(dX^{n+m} \wedge dX^j - dX^j \wedge dX^{n+m}) = E'. \end{aligned}$$

From this

$$(E'_{AB}) = \begin{pmatrix} \partial_{[k}g_{i]m}X^{n+m} & -g_{ik} \\ g_{ik} & 0 \end{pmatrix}. \tag{8}$$

PROPOSITION 4. The skew-symmetric tensors E'_{AB} (8) in $T(V_n)$ and E_{AB}(7) in $T^c(V_n)$ defined by the exterior differential of the fundamental form p are the corresponding tensors with respect to the diffeomorphism f.

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Rymano daugdaros liečiamojo ir koliečiamojo pluoštų diferencialinės geometrijos klausimu

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Nagrinėjama Rymano daugdara V_n, jos liečiamasis pluoštas $T(V_n)$ ir koliečiamasis pluoštas $T^c(V_n)$. Difeomorfizmas $f : T(V_n) \to T^c(V_n)$ apibrėžiamas kanoniniu V_n izomorfizmu. Darbe nagrinėjami tam tikrų daugdaros V_n tenzorių liftai daugdarose $T(V_n)$ ir $T^c(V_n)$ bei ieškomas ryšys tarp tų liftų.

Įrodyta teorema: difeomorfizmas f V_n metrikos pilną liftą daugdaroje $T(V_n)$ atvaizduoja į tos metrikos pratęsimą (Jano-Patersono prasme) daugdaroje $T^c(V_n)$.