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Algebraic version of convex combination patches

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1. Introduction

Sabin [8], [9] and Hosaka & Kimura [4] introduced 3-, 5- and 6-sided patches suitable for a smooth joining to rectangular Bézier patches. Their approach was generalized by Zheng and Ball in [10]. Loop and DeRose introduced in [7] Sabin and Hosaka–Kimura like patches with arbitrary number of sides. In [5] and [6] are given more efficient methods for a construction of this type of patches. A method, presented in this paper, can be treated as an algebraic simulation of a convex combination patches of Gregory & Charrot (see [1], [2], [3]). New method produces in many important cases more efficient patches as in [5]. Algebraic methods are also used for a construction of multisided Coons and Steiner like patches.

This paper is organized as follows. In Section 2 is described a plotting of multisided patches. Algebraic simulation of convex combination method is presented in Section 3. Multisided generalization of Coons and Steiner patches is given in Section 4. In this paper we do not give the proofs.

2. Domain and plotting of multisided patches

Let D be a regular m-gon in (x, y)-plane with the vertices $V_s = (\cos s\alpha; \sin s\alpha), 0 \leqslant s \leqslant m-1$. A point of an intersection of the lines $\overline{V_{s-1}V_s}$ and $\overline{V_{s+1}V_{s+2}}$ is denoted by K_s. We set

$$l_s = -x\cos(s\alpha + \frac{\alpha}{2}) - y\sin(s\alpha + \frac{\alpha}{2}) + \cos\frac{\alpha}{2}, \ 0 \leqslant s \leqslant m-1. \tag{1}$$

It is easy to check that an equation $l_s = 0$ defines a line $\overline{V_sV_{s+1}}$.

Suppose: $\mathcal{L}$ is a graph having a symmetry group of a regular m-gon; there are fixed basis functions $f_q, q \in \mathcal{L}$.

DEFINITION 1. *A parametric rational m-patch over domain D is a map* $F : D \to \mathbb{R}^3$ *defined by the formula*

$$F(p) = \frac{\sum_{q\in\mathcal{L}} w_q P_q f_q(p)}{\sum_{q\in\mathcal{L}} w_q f_q(p)}. \tag{2}$$

The points P_q are called control points of the patch and the numbers w_q are their weights. Geometrically a patch is understood as the image $F(D)$.

Following widely used method for plotting of multisided patches is convenient in a practical implementation: a domain D is subdivided into quadrangular regions: bilinear parametrizations of these regions are composed with a map F, defining a patch; the compositions are converted to the rational tenzor product form; patch is plotted as a collection of rational rectangular Bézier patches. Various type of multisided patches will be compaired from a point of view of a just described method: a patch is more efficient (compaired with another one) if it can represented as a collection of Bézier patches of lower bidegree. This efficience in many cases can be achieved taking care on the behaviour of the basis functions at the points K_s. Desired behaviour is formalized using some standart concepts from algebraic geometry.

DEFINITION 2. (1) *A polynomial $f(x,y)$ has at a point $(x_0;y_0)$ zero of multiplicity μ if it vanishes at $(x_0;y_0)$ together with all partial derivatives up to the order $\mu-1$ and at least one partial derivative of order μ does not vanish;*
(2) *a point p is base point of multiplicity μ of a rational map*

$$G:(x,y)\mapsto (g_1(x,y)/g_0(x,y), g_2(x,y)/g_0(x,y), g_3(x,y)/g_0(x,y))$$

if all polynomials $g_i(x,y)$ have at p zero of multiplicity at least μ.

Let $M_s=(V_s+V_{s+1})/2$ and O is a center of a regular m-gon, $m\geqslant 5$. Subdividing a domain D into m quadrangles $V_sM_sOM_{s-1}$, and taking their suitable rational bilinear parametrizations we get the following estimation for the bidegrees of Bézier patches (see [5]).

PROPOSITION 1. *If a rational map G is of degree n and K_s, $s=0,1,\ldots,m-1$, are its base points of multiplicity μ, then a patch $G(D)$ can be represented as a collection of m rational rectangular Bézier patches of bidegree $(n-\mu,n-\mu)$.*

For a plotting of triangular and rectangular patches ($m=3,4$) the standart methods are used.

3. Sabin and Hosaka–Kimura like patches

Control points and weights of m-sided Sabin and Hosaka–Kimura like surface patches are labeled by the triples $(s,0,j)$, $(s,1,k)$ $0\leqslant s\leqslant m-1$, $0\leqslant j\leqslant n$, $1\leqslant k\leqslant n-1$, where $(s-1,0,n)=(s,0,0)$ and $(s-1,1,n-1)=(s,1,1)$. (Index s is treated in a cyclic fashion; degree of the boundary curves is n.) A set of these triples is denoted by $\mathcal{H}^n$. In [5] they are called SHK-patches and their formal definition is given. SHK-patches posses following nice property: for each boundary Bézier curve with control points P_{s0j} and weights w_{s0j}, $j=0,1,\ldots,n$, a patch can be so reparametrized, that crossderivative along this curve is the same as crossderivative of rectangular Bézier patch with two layers of control points (and corresponding weights) $P_{s,0,0}, P_{s,0,1},\ldots,P_{s,0,n}$ and $P_{s-1,0,n-1}, P_{s,1,1},\ldots,P_{s+1,1,1}, P_{s+1,0,1}$. It

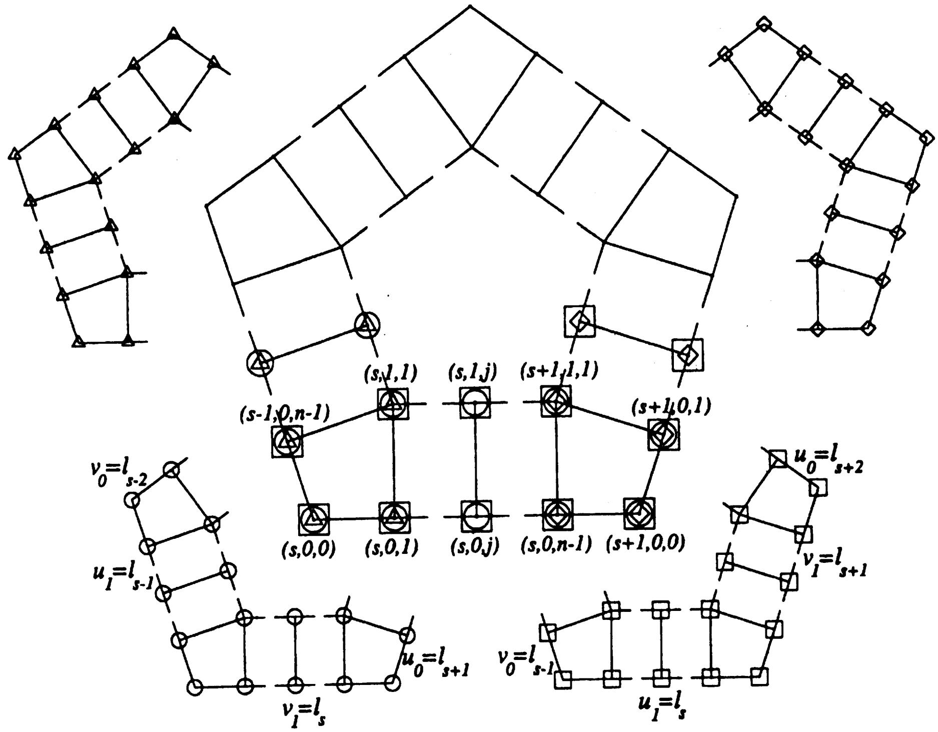

Fig. 1. Graph of SHK-patches and principle of algebraic simulation.

allows to join smoothly SHK-patches with adjacient Bézier patches and to build spline surfaces of arbitrary topology.The auxilary functions g_{sij}, $s = 0, 1, \ldots, m-1$, $0 \leqslant i, j \leqslant n$, are defined by the formula

$$g_{sij} = \binom{n}{i}\binom{n}{j} l_{s-2}^{n-i} l_s^i l_{s+1}^{n-j} l_{s-1}^j (l_{s+1} l_{s+2} \ldots l_{s-2})^2. \tag{3}$$

For $q = (s, i, j)$, $q \in \mathcal{H}^n$ we set

$$\begin{aligned}
f_{s,0,0} &= g_{s,0,0} + g_{s-1,0,n} + g_{s+1,n,0};\\
f_{s,0,1} &= g_{s,0,1} + g_{s-1,1,n} + g_{s+1,n-1,0};\\
f_{s,0,n-1} &= g_{s,0,n-1} + g_{s-1,n-1,n} + g_{s+1,1,0};\\
f_{s,0,j} &= g_{s,0,j} + g_{s+1,n-j,0}, \text{if } n \geqslant 4,\ 2 \leqslant j \leqslant n-2;\\
f_{0,1,1} &= g_{0,1,1} + g_{1,1,1} + \ldots + g_{m-1,1,1}, \text{ if } n = 2;\\
f_{s,1,1} &= g_{s,1,1} + g_{s-1,1,n-1} + g_{s+1,n-1,1}, \text{ if } n \geqslant 3;\\
f_{s,1,j} &= g_{s,1,j} + g_{s+1,n-j,1}, \text{if } n \geqslant 4,\ 2 \leqslant j \leqslant n-2.
\end{aligned} \tag{4}$$

Theorem 1. *If the basis functions are given by the formulas* (4) *a patch, defined by the formula* (2), *where* $\mathcal{L} = \mathcal{H}^n$, *is* SHK*-patch of order* n.

Comparison with other patches. Presented here construction of SHK-patches can be considered as an algebraic simulation of convex combination method of Gregory–Charrot (see [1], [3]). So let us compaire both approaches. If boundary curves of m-sided patch are Bézier curves of degree n and surrounding surfaces are rectangular Bézier patches, convex combination method produces a rational map of degree $nm + 2m - 4$ with the base points K_s of multiplicity $n + 2$. It follows from Proposition 1 that constructed surface is a collection of Bézier patches of bidegree (a, a), where $a = (m-1)n + 2m - 6$. A patch from Theorem 1 is defined by a rational map of degree $2n + 2m - 4$ with the base points K_s of multiplicity 4. So this patch is a collection of Bézier patches of bidegree (b, b), where $b = 2n + 2m - 8$. This means algebraic simulation produces more efficient patches.

In [5] constructed SHK-patches can be represented as a collection of Bézier patches of bidegree (c, c), where $c = (m-3)n+1$. They are more efficient as presented in [7], [8], [10]. In [6] most efficient 5- and 6-sided patches are developed. Compairing the estimations b and c we see that if $m \geqslant 7, n \geqslant 3$, the algebraic simulation method produces the patches of lower degree as derived in [5].

4. Coons like multisided patches

Let $\mathcal{L}^n$ be a set of the pairs (s, j), $s = 0, 1, \ldots, m-1, j = 0, 1, \ldots, n$, where $(s-1, n) = (s, 0)$ (index s is treated in a cyclic fashion). Obviously $\mathcal{L}^n$ can be considered as a subgraph of the graph $\mathcal{H}^n$ with the second index i equal to zero. The auxilary functions g_{sj}, $s = 0, 1, \ldots, m-1$, $0 \leqslant i, j \leqslant n$, are defined by the formulas

$$\begin{aligned} g_{s0} &= l_{s+1} l_{s+2} \ldots l_{s-2}; \\ g_{sj} &= l_{s+1}^{n-j} l_{s+2} \ldots l_{s-2} l_{s-1}^{j}, \text{ if } 1 \leqslant j \leqslant n-1. \end{aligned} \tag{5}$$

Let $a = 2 \sin\alpha \sin(\alpha/2)$, $b_j = -\binom{n-1}{j}/a^{n-1}$, $c_j = \binom{n}{j}/a^{n-1}$, $1 \leqslant j \leqslant n-1$. For $q = (s, j)$, $q \in \mathcal{L}^n$, the functions f_q are defined by the formulas

$$\begin{aligned} f_{s,0} &= g_{s,0} + \textstyle\sum_{j=1}^{n-1} b_j (g_{s,j} + g_{s-1,n-j}), \\ f_{s,j} &= c_j g_{s,j}, \ 1 \leqslant j \leqslant n-1. \end{aligned} \tag{6}$$

Theorem 2. *Let $\mathcal{L} = \mathcal{L}^n$. Suppose the basis functions f_q are given by the formulas* (6). *Then boundary curves of m-sided patch defined by the formula* (2) *are rational Bézier curves with the control points $P_{s0}, P_{s1}, \ldots, P_{sn}$ and weights $w_{s0}, w_{s1}, \ldots, w_{sn}$, $s = 0, 1, \ldots, m-1$.*

In case $m = 3, 4$ it is convenient to change the notations of the functions l_i to the standart notations of triangular or tenzor product patches. By λ_0, λ_1, λ_2 are denoted the barycentric coordinates respect to the points V_0, V_1, V_2. For any triangle with the vertices V_0, V_1, V_2 we set $l_0 = \lambda_2, l_1 = \lambda_0, l_2 = \lambda_1$. For a standart unit rectangle in (x, y)-plane we set $l_0 = y$, $l_1 = 1-x$, $l_2 = 1-y$, $l_3 = x$. In case of changed notations we assume (for both cases) $a = 1$. Theorem 2 remains true after these changes.

For $m = 4$ the patches from Theorem 2 coincides with the classical Coons patches. It means constructed patches can be treated as multisided Coons patches, if boundary consists of rational Bézier curves of the same degree. We briefly describe some other indications, that derived patches are right generalization.

- By R_k, $1 \leqslant k \leqslant m(m-3)/2$, are denoted the intersections of the sides of domain m-gon, distinct from K_s. Basis polynomials $f_q, q \in \mathcal{L}^n$, are of degree $m+n-3$ and have zeros at the points R_k. If $n = 2$, there are exactly $2m$ independent polynomials with these properties. Therefore, if boundary curves are conics these patches can be treated as the multisided versions of Steiner surfaces.
- For $m = 3, 4$ the basis functions sum to 1. It is impossible if $m \geqslant 5$, since there are base points in affine plane. But the sum of basis functions in this case is possibly simple and equal to the product of the equations of the $[(m-3)/2]$ concentric circles, going through all finite points R_k.
- Degree of the patches from Theorem 2 is less compaired with the degree of the patches, produced using convex combination method. Moreover, $m+n-3$ seems to be lowest possible degree.

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Algebrinė iškilių skiaučių kombinacijų versija

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Darbe pateikiama algebrinė iškilių skiaučių kombinacijų versija. Taip pat apibendrinami Kunso ir Šteinerio paviršiai daugiakampėms skiautėms. Skiaučių konstravimui naudojama algebrinėje geometrijoje gerai žinoma bazinių taškų technika, kuri geometriniame modeliavime nėra paplitusi.