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On decidability of a fragment of intuitionistic predicate logic

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1. Introduction

The aim of this paper is to prove the decidability in first order intuitionistic logic of the following class of sequents: $\Sigma, \forall\Delta \to P$, where Σ (called parametrical part) consists of elementary formulas with different predicate symbols; P is an elementary formula; $\forall\Delta = \forall x_1(E_{11}(x_1) \supset E_{12}(\bar{f}_1(x_1)))$, …, $\forall x_n(E_{n1}(x) \supset E_{n2}(\bar{f}_n(x_n)))$, where $E_{i1} \neq E_{j2}$ $(1 \leqslant i, j \leqslant n)$ (looping-free condition); E_{ik} $(1 \leqslant i \leqslant n,\ 1 \leqslant k \leqslant 2)$ is a one-place predicate symbol; $E_{i1}(x_i)$ (called premise formula) – an elementary formula without functional symbols; $E_{i2}(\bar{f}_i(x_i))$ (called conclusion formula) – an elementary formula with functional symbols; $\bar{f}_i(x_i) = f_{i1}(f_{i2} \dots (f_{il}(x_i)) \dots)$ $(l \geqslant 1)$; f_{ij} $(1 \leqslant j \leqslant l)$ – one-place functional symbol. All premise and conclusion formulas contain different predicate symbols. Such sequents will be called KP-sequents. The requirement that KP-sequent contains only one-place predicate and functional symbols is not essential, it enables us to simplify notations and technical details. The requirements that parametrical part of KP-sequents consists of elementary formulas with different predicate symbols and that premise formulas not contain functional symbols also are not essential, they enable us to simplify the proposed decision procedure. But looping-free condition is essential, it enables us to construct contraction-free calculus for KP-sequents and this calculus is the main step to prove the decidability of KP-sequents.

2. Description of calculi G_0, G_1, G_2

In this section the initial calculus G_0 and auxiliary calculi G_1, G_2 will be introduced.

DEFINITION 1 (sequent). A sequent is an expression of the form $\Gamma \to \Theta$, where Γ is arbitrary multiset, Θ is either empty word or any formula.

Now we dropp the requirement from the definition of KP-sequents different predicate symbols condition which is non-essential in getting contraction-free calculus.

DEFINITION 2 (KP_1-sequent). Definition of KP_1-sequents is obtained from definition of KP-sequents by dropping different predicate symbols condition on parametrical part and on premise and coclusion formulas.

DEFINITION 3 (calculus G_0). The calculus is defined by the following postulates.

Axiom: $\Gamma, E \to E$, where E is an elementary formula.

Rules:

$$\frac{E_1, E_2, \Gamma \to P}{E_1, E_1 \supset E_2, \Gamma \to P}\,(\supset_e \to) \qquad \frac{A(t), \forall x A(x), \Gamma \to P}{\forall x A(x), \Gamma \to P}\,(\forall \to),$$

where E_1, E_2 are elementary formulas, $A(x) = E_1(x) \supset E_2(\bar{f}(x))$, t – term free for x in $A(x)$ (see e.g. [2]). All derivation in G_0 are constructed in a linear form.

Theorem 1. *Let S be a KP_1-sequent then the calculus G_0 is sound and complete.*

Proof. Analogously as in [1].

DEFINITION 4 (calculus G_1). The calculus G_1 is obtained from calculus G_0 replacing rule $(\forall \to)$ by the following one:

$$\frac{E_1(t), \forall x(E_1(x) \supset E_2(\bar{f}(x))), E_2(\bar{f}(t)), \Gamma \to P}{E_1(t), \forall x(E_1(x) \supset E_2(\bar{f}(x))), \Gamma \to P}\,(\forall^+ \to).$$

The formula $\forall x(E_1(x) \supset E_2(\bar{f}(x)))$ is called the main formula of $(\forall^+ \to)$; $E_1(t)$ – the elementary main formula (in short: e-main formula) of $(\forall^+ \to)$; $E_2(\bar{f}(t))$ – the side formula of $(\forall^+ \to)$. Analogously is defined the main, e-main and side formula of $(\supset_e \to)$.

Lemma 1. *The rule $(\forall^+ \to)$ is admissible in G_0.*

Proof. Using rules $(\supset_e \to)$, $(\forall \to)$.

Lemma 2. $G_1 \vdash S \Rightarrow G_0 \vdash S$.

Proof. Follows from Lemma 1.

DEFINITION 5 (calculus G_2). The calculus G_2 is obtained from the calculus G_0 by adding the rule $(\forall^+ \to)$.

Lemma 3. *The structural rule weakening, i.e.* $\frac{\Gamma \to P}{\Pi, \Gamma \to P}\,(W)$ *is admissible in calculus* $I_0 \in \{G_0, G_1, G_2\}$.

Proof. By evident induction on the height of derivation of the sequent $\Gamma \to P$.

Lemma 4. *The rule $(\supset_e \to)$ is invertible in calculus $I \in \{G_0, G_1, G_2\}$.*

Proof. By evident induction on the height of the conclusion of $(\supset_e \to)$.

Lemma 5. *The structural rule* $\dfrac{E, E, \Gamma \to P}{E, \Gamma \to P}$ $(C_e \to)$ *is admissible in calculus* $I \in \{G_0, G_1, G_2\}$.

Proof. By evident induction on the height of derivation of the sequent $E, E, \Gamma \to P$.

Lemma 6. *Let* S *be a* KP_1*-sequent, then* $G_2 \overset{V}{\vdash} S \Rightarrow G_1 \overset{V^*}{\vdash} S$.

Proof. Let $V(\forall \to)$ be the number of applications of $(\forall \to)$ in the given derivation V. The proof is carried out by $V(\forall \to)$. If $V(\forall \to) = 0$, then $V = V^*$. Let $V(\forall \to) > 0$. Let us consider the highest application of $(\forall \to)$ in V.

$$V_1 \left\{ \begin{array}{c} \dots \\ \hline A(t), \forall x A(x), \Gamma \to P \\ \hline \forall x A(x), \Gamma \to P \end{array} \right. \begin{array}{l} (i) \\ (\forall \to), \end{array}$$

where $A(x) = E_1(x) \supset E_2(\bar{f}(x))$. To found the induction step the induction on $h(V_1)$ is used.

DEFINITION 6 (calculus G_3). The calculus G_3 is obtained from calculus G_1 by dropping rule $(\supset_e \to)$.

Lemma 7. *Let* S *be* KP_1*-sequent, then* $G_0 \vdash S \Rightarrow G_3 \vdash S$.

Proof. Let $G_0 \overset{V_1}{\vdash} S$, then by definition of G_2 we have that $G_2 \overset{V_2}{\vdash} S$. From Lemma 6 we have that $G_1 \overset{V_3}{\vdash} S$. Since S is KP_1-sequent then V_3 does not contain the applications of the rule $(\supset_e \to)$. Therefore $G_3 \vdash S$.

3. Contraction-free calculus G_4

In this section we shall show that duplication of the main formula in $(\forall^+ \to)$ is not necessary for KP_1-sequents.

DEFINITION 7 (calculus G_4). The calculus G_4 is obtained from calculus G_3 replacing the rule $(\forall^+ \to)$ by the following one:

$$\frac{E_1(t), E_2(\bar{f}(t)), \Gamma \to P}{E_1(t), \forall x(E_1(x) \supset E_2(\bar{f}(x))), \Gamma \to P} (\forall^* \to).$$

To prove that $G_3 \vdash S \Rightarrow G_4 \vdash S$ let us introduce the following relationship relations.

DEFINITION 8 (ancestor, descedant, trace). Let us consider an application of $(\forall^* \to)$. The side formula $E_2(\bar{f}(t))$ is called descedant of e-main formula $E_1(t)$ and the e-main formula $E_1(t)$

is called ancestor of the side formula $E_2(\bar{f}(t))$. We say that a formula $E_{i+p+1}(t_{i+p+1})$ is the descedant of the formula $E_i(t_i)$ (and $E_i(t_i)$ is the ancestor of the formula $E_{i+p+1}(t_{i+p+1})$ if the list (called trace of the formula $E_i(t_i)E_i(t_i)E_{i+1}(t_{i+1}), \ldots, E_{i+p}(t_{i+p}), E_{i+p+1}(t_{i+p+1})$ satisfies the following condition: $\forall \rho$ $(0 \leqslant \rho \leqslant p)$ $E_{i+\rho}(t_{i+\rho})$ is e-main and $E_{i+\rho+1}(t_{i+\rho+1})$ is the side formula of $(\forall^* \rightarrow)$ with the same main formula.

DEFINITION 9 (δ-application of $(\forall^* \rightarrow)$, $\delta \in \{\Sigma, \Pi\}$, Π-regular applications of $(\forall^+ \rightarrow)$, regular derivation). Let $S = \Sigma, \forall\Delta \rightarrow P$ be KP_1-sequent, let us consider derivation V of S in calculus G_3. The applications of $(\forall^+ \rightarrow)$ is called Σ-application of $(\forall^+ \rightarrow)$ if the e-main formula of $(\forall^+ \rightarrow)$ is an elementary formula friom Σ and is called Π-application of $(\forall^+ \rightarrow)$ if the e-main formula of $(\forall^+ \rightarrow)$ is an elementary formula obtained during the process of the derivation V. An application of $(\forall^+ \rightarrow)$ is called Π-regular if above Π-application of $(\forall^+ \rightarrow)$ is only Π-applications of $(\forall^+ \rightarrow)$. A derivation of KP_1-sequent in G_3 is called regular if all Π-applications of $(\forall^+ \rightarrow)$ are Π-regular.

Lemma 8. *Let S be KP_1-sequent, then $G_3 \overset{V}{\vdash} S \Rightarrow G_3 \overset{V^*}{\vdash} S$, where V^* is the regular derivation.*

Proof. By interchanging the Σ-applications with Π-applications of $(\forall^+ \rightarrow)$.

Lemma 9. *Let $G_3 \overset{V}{\vdash} S$, where V is regular derivation of KP_1-sequent. Let $E_i(t_i)$ be e-main formula of Π-application of $(\forall^+ \rightarrow)$ in V. Let us consider the trace of the formula $E_i(t_i)$, i.e. the list of elementary formulas $E_i(t_i), E_{i+1}(t_{i+1}), \ldots, E_{i+p}(t_{i+p})$. Than $\forall km$ $(0 \leqslant k,\ m \leqslant p)$ $E_k \neq E_p$, i.e. the trace consist of elementary formulas with different predicate symbols.*

Proof. Induction on the lenght of the trace of formula $E_i(t_i)$.

Lemma 10. *Let S be KP_1-sequent, then $G_3 \overset{V}{\vdash} S \Longrightarrow G_4 \overset{V^*}{\vdash} S$, where V is regular derivation.*

Proof. Let $E_2(\bar{f}(t))$, $E_2(\bar{f}(q))$ be conclusion formulas from the formula $\forall x(E_1(x) \supset E_2(\bar{f}(x)))$ and t, q be the values of the variable x, then the pair $E_2(\bar{f}(t))$, $E_2(\bar{f}(q))$ is called the singular pair. Let S^* be the axiom of V and $|S^*|$ be the number of the singular pairs in V. The proof is carried out by induction on $|S|$. We can assume that all applications of $(\forall^+ \rightarrow)$ are essential.

Let $|S^*| = 0$, then $V = V^*$.

Let $|S^*| > 0$. Let us consider the origin of the highest singular pair $E_2(\bar{f}(t))$, $E_2(\bar{f}(q))$ in V:

$$V' \left\{ \begin{array}{c} \dfrac{\Sigma, \forall x A, E_2(\bar{f}(t)), E_2(\bar{f}(q)), E_1(q), \Gamma \to P}{\Sigma, \forall x A, E_2(\bar{f}(t)), E_1(q), \Gamma \to P} (\forall^+ \to) \\ \vdots \\ S \end{array} \right.$$

where $\forall x A(x) = \forall x(E_1(x) \supset E_2(\bar{f}(x)))$ and $E_1(q)$ $(E_2(\bar{f}(q)))$ is e-main (side, correspondingly) formula of the application of $(\forall^+ \to)$. According to the loop-free condition on KP_1-sequent we have that $E_1 \neq E_2$.

Let us consider the following cases:

1. $E_2(\bar{f}(q))$ is not the ancestor of the main formula of the axiom S^*. In this case we can dropp the application of $(\forall^+ \to)$ and the formula $E_2(\bar{f}(q))$ reducing the $|S^*|$.

2. $E_2(\bar{f}(q))$ is the ancestor of the main formula of the axiom S^*.

2.1. $E_2(\bar{f}(t))$ is the e-main formula of $(\forall^+ \to)$ in the part V' of the given derivation V. Since all applications of $(\forall^+ \to)$ are essential, $E_2(\bar{f}(t))$ is the ancestor of the main formula of axiom S^*, and also $E_2(\bar{f}(t))$ is the ancestor of the formula $E_2(\bar{f}(q))$ (which, by the asumption of the case 2), is the ancestor of the main formula of the axiom S^*. It means that there exists the trace of the formula $E_2(\bar{f}(t))$ containing the $E_2(\bar{f}(q))$. But this is not possible by the Lemma 9.

2.2. $E_2(\bar{f}(t))$ is not e-main formula of $(\forall^+ \to)$ in V'. In this case we can dropp the formula $E_2(\bar{f}(t))$ reducing the induction parameter $|S^*|$.

REMARK 1. The loop-free condition is the essential in proving the Lemma 10. Indeed, let $S = P(c), \forall y(P(y) \supset R(y)), \forall z(R(z) \supset P(f(z))) \to R(f(c))$. Then it is easy to verify that $G_3 \vdash S$, but $G_4 \nvdash S$.

Lemma 11. *Let S be KP_1-sequent then $G_0 \vdash S \Longrightarrow G_4 \vdash S$.*

Proof. Follows from Lemmas 7, 10.

Theorem 2. *Let S be KP_1-sequent, then the calculus G_4 is sound and complete.*

Proof. Follows from Lemma 11 and Theorem 1.

4. Description of the decision procedure for KP-sequents

Now we shall apply the contraction-free calculus G_4 for getting the decision procedure for KP-sequents (which satisfy not only loop-free condition but also different predicate symbols condition).

Lemma 12. *Let* $S - \Sigma, \forall x(E_1(x) \supset E_2(\bar{f}(x))), \forall\Delta \to P$ *and let* $G_4 \overset{V}{\vdash} S$. *Then* (1) *either* $G_4 \vdash \Sigma, \forall\Delta \to P$ *if* $E_1(t) \notin \Sigma$, *or* (2) $G_4 \vdash \Sigma, E_2(\bar{f}(t)), \forall\Delta \to P$ *if* $E_1(t) \in \Sigma$.

Proof. By induction on $h(V)$.

Lemma 13. *The calculus* G_4 *is decidable for* KP*-sequents.*

Proof. Follows from Lemma 12.

References

[1] R. Dyckhoff, Contraction-free sequent calculi for intuitionistic logic, *Journal of Symbolic Logic*, **57**, 795–807 (1992).
[2] S.C. Kleene, *Introcduction to meta-mathematics*, D. Van Nostrand Company, New York (1952).

Apie išsprendžiamą intuicionistinės kvantorinės logikos fragmentą

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Straipsnyje sukonstruotas beciklis skaičiavimas intuicionistinės kvantorinės logikos fragmentui. Įrodytas sukonstruoto skaičiavimo korektiškumas ir pilnumas. Remiantis tuo įrodytas nagrinėjamo intuicionistinės kvantorinės logikos fragmento išsprendžiamumas.