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Refinement of the L_1 estimate in the central limit theorem for m-dependent random fields

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Denote $\mathbb{Z}^d = \{a = (a_1, a_2, \dots, a_d),\ a_i \in \mathbb{Z},\ i = 1, 2, \dots, d\}$, $||a|| = \max\limits_{1\leqslant i\leqslant d} |a_i|$, where $\mathbb{Z}$ is the set of integers. Define the distance between the sets $V_1, V_2 \subset \mathbb{Z}^d$ as follows: $d(V_1, V_2) = \inf\{||a-b|| :\ a \in V_1,\ b \in V_2\}$. Denote by $\mathcal{F}_V$ the σ-algebra of events generated by random variables (r.v.s) $\{X_a,\ a \in V\}$.

The random field $\{X_a,\ a \in \mathbb{Z}^d\}$ is said to be m-dependent, if for any $V_1, V_2 \subset \mathbb{Z}^d$ the σ-algebras $\mathcal{F}_{V_1}$ and $\mathcal{F}_{V_2}$ are independent whenever $d(V_1, V_2) > m$.

In the sequel $V,\ V \subset \mathbb{Z}^d$, is any fixed nonempty finite set.

Denote

$$S_V = \sum_{a\in V} X_a, \quad B_V^2 = \mathbf{E}S_V^2, \quad Z_V = S_V/B_V, \quad A_a = X_a/B_V,$$

$$\overline{L}_r(t) = \sum_{a\in V} \mathbf{E}|A_a|^r \mathbb{I}_{\{|A_a|\leqslant t\}}, \quad \overline{\overline{L}}_r(t) = \sum_{a\in V} \mathbf{E}|A_a|^r \mathbb{I}_{\{|A_a|>t\}},$$

$$\Delta_V(x) = \mathbf{P}\{Z_V < x\} - \Phi(x), \quad \Phi(x) = (2\pi)^{-1/2}\int\limits_{-\infty}^{x} e^{-u^2/2}du,$$

$$||\cdot||_1 = \int\limits_{-\infty}^{\infty} |\cdot|dx, \quad L_r = \sum_{a\in V} \mathbf{E}|A_a|^r,$$

where $t > 0$, $B_V > 0$ and $\mathbb{I}_A$ is the indicator of event A.

By $C(\cdot)$ we denote positive finite constants (in general, different) depending only on the quantities indicated in the parentheses.

In the present paper, we have got a precise upper bound of the quantity $||\Delta_V(x)||_1$ for real m-dependent random fields under the finiteness of the second order moments of summands X_a (see [8], Theorem 6).

The following statement is valid.

Theorem 1. *Suppose that $\{X_a,\ a \in \mathbb{Z}^d\}$, $d \geqslant 1$, is a real m-dependent random field with*

$$\mathbf{E}X_a = 0 \quad and \quad \mathbf{E}X_a^2 < \infty \quad for\ all \quad a \in V. \tag{1}$$

Then for all $m \geqslant 0$ *and* $t > 0$

$$||\Delta_V(x)||_1 \leqslant C(d)\Big\{\overline{\overline{L}}_1(t) + (m+1)^d\overline{\overline{L}}_2(t) + (m+1)^{2d}\overline{L}_3(t) + (m+1)^{3d}\overline{L}_4(t)\Big\}. \tag{2}$$

In addition, if $\mathbf{E}|X_a|^s < \infty$ *for some* s, $2 < s \leqslant 3$, *and all* $a \in V$, *then for all* $m \geqslant 0$

$$||\Delta_V(x)||_1 \leqslant C(d)(m+1)^{d(s-1)}L_s. \tag{3}$$

Note that the estimates of $||\Delta_V(x)||_1$ are obtained in terms of Lyapunov fraction, and estimate (3) is exact in the independent case ($m = 0$).

In order to prove Theorem 1 we need the following statement.

Theorem A [8]. *Assume that* $\{X_a,\ a \in \mathbb{Z}^d\}$, $d \geqslant 1$, *is a real* m*-dependent random field with*

$$\mathbf{E}X_a = 0 \quad \text{and} \quad \mathbf{E}X_a^4 < \infty \quad \text{for all} \quad a \in V. \tag{4}$$

Then for all $m \geqslant 0$

$$||\Delta_V(x)||_1 \leqslant C(d)\Big\{(m+1)^{2d}L_3 + (m+1)^{3d}L_4\Big\}. \tag{5}$$

Recall (see [8]) that to obtain estimates of $||\Delta_V(x)||_1$ where m-dependent random fields are summed, we expand $\Delta_V(x)$ by the local partitioning method into summands which include the mixed moments of 'close' blocks composed of the initial random field (as usual, each summand of this kind includes three or four blocks).

In order to expand $\Delta_V(x)$ and estimate $||\Delta_V(x)||_1$ we use the properties of the linear differential equation

$$f'(y) - yf(y) = \mathrm{I\!I}_{(-\infty,x)}(y) - \Phi(x), \tag{6}$$

proposed by Stein (1972), and its solution (see Erickson (1974), Ho and Chen (1978)).

Proof of Theorem 1. Estimate (3) is proved in [8]. We now prove estimate (2).

Truncation. Let a real random field $\{X_a,\ a \in \mathbb{Z}^d\}$, $d \geqslant 1$, satisfy (1). Truncate r.v.s for all $a \in V$ on the level $t > 0$. Denote

$$\overline{A}_a = A_a \mathrm{I\!I}_{\{|A_a| \leqslant t\}},\ \overline{A}_a^{(0)} = \overline{A}_a - \mathbf{E}\overline{A}_a,\ \overline{Z}_V^{(0)} = \sum_{a\in V}\overline{A}_a^{(0)},\ (\overline{B}_V^{(0)})^2 = \mathbf{E}(\overline{Z}_V^{(0)})^2,$$

$$\overline{\overline{A}}_a = A_a \mathrm{I\!I}_{\{|A_a| > t\}},\quad \overline{\overline{A}}_a^{(0)} = \overline{\overline{A}}_a - \mathbf{E}\overline{\overline{A}}_a.$$

It is known that for any real r.v. ξ and η

$$\left\|\mathbf{P}\{\xi < x\} - \mathbf{P}\{\eta < x\}\right\|_1 \leqslant \mathbf{E}|\xi - \eta|. \tag{7}$$

Let $\overline{B}_V^{(0)} > 0$. Using (7), we get that

$$\left\|\mathbf{P}\{Z_V < x\} - \mathbf{P}\{\overline{Z}_V^{(0)} < x\}\right\|_1 \leqslant \mathbf{E}\left|\overline{\overline{Z}}_V^{(0)}\right| \tag{8}$$

and

$$\left\|\Phi(x/\overline{B}_V^{(0)}) - \Phi(x)\right\|_1 \leqslant \mathbf{E}|\mathcal{N}|\left|1 - \overline{B}_V^{(0)}\right|; \tag{9}$$

where $\mathcal{N}$ is a standard normal r.v. with the distribution function $\Phi(x)$.

Note that

$$\begin{aligned}\left\|\mathbf{P}\{\overline{Z}_V^{(0)} < x\} - \mathbf{P}\{\mathcal{N} < x(\overline{B}_V^{(0)})^{-1}\}\right\|_1 = \overline{B}_V^{(0)} \ \Big\| \ &\mathbf{P}\{\overline{Z}_V^{(0)} < x\overline{B}_V^{(0)}\} \\ &- \mathbf{P}\{\mathcal{N} < x\}\Big\|_1.\end{aligned} \tag{10}$$

Therefore, from (8)–(10) we get that, if $\overline{B}_V^{(0)} > 0$, then for all $t > 0$

$$\begin{aligned}||\Delta_V(x)||_1 \leqslant\ & 2\overline{\overline{L}}_1(t) + \sqrt{2/\pi}\left|1 - (\overline{B}_V^{(0)})^2\right| \\ &+ \overline{B}_V^{(0)}\left\|\mathbf{P}\{\overline{Z}_V^{(0)} < x\overline{B}_V^{(0)}\} - \Phi(x)\right\|_1.\end{aligned} \tag{11}$$

It is easy to see that

$$\begin{aligned}\mathbf{E}Z_V^2 - \mathbf{E}(\overline{Z}_V^{(0)})^2 =\ & \sum_{a\in V}\{\mathbf{E}\overline{\overline{A}}_a^2 + (\mathbf{E}\overline{\overline{A}}_a)^2\} \\ &+ \sum_{a\in V}\sum_{\substack{b\in V\\ a\neq b}}\{\mathbf{E}\overline{\overline{A}}_a^{(0)}\overline{\overline{A}}_b^{(0)} + 2\mathbf{E}\overline{A}_a^{(0)}\overline{\overline{A}}_b^{(0)}\}.\end{aligned} \tag{12}$$

Note that truncation inequality (11) and equality (12) hold for any dependence of the random field $\{X_a,\ a \in \mathbb{Z}^d\},\ d \geqslant 1$.

By virtue of the m-dependence, Hölder's inequality, and the fact that $\mathbf{E}\left|\overline{\overline{A}}_a^{(0)}\right| \leqslant 2t^{-1}\mathbf{E}\overline{\overline{A}}_a^2$ for all $a \in V$, we get from (12) that

$$\left|1 - (\overline{B}_V^{(0)})^2\right| \leqslant 9(2m+1)^d\overline{\overline{L}}_2(t) \equiv \varepsilon. \tag{13}$$

Thanks to (7), we have that $||\Delta_V(x)||_1 \leqslant \sqrt{2}$. Therefore, in estimating $||\Delta_V(x)||_1$ we assume, without loss of generality, that $\varepsilon \leqslant 1/2$ (then we have $\frac{1}{2} \leqslant (\overline{B}_V^{(0)})^2 \leqslant \frac{3}{2}$). Estimate (2) now follows from (11), (13), and Theorem A.

Note that estimate (3) follows from estimate (2) for $t = (m+1)^{-d}$. Theorem 1 is proved.

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L_1 įverčio patikslinimas centrinėje ribinėje teoremoje m-priklausomiems atsitiktiniams laukams

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Gautas L_1 įverčio patikslinimas centrinėje ribinėje teoremoje realiems m-priklausomiems atsitiktiniams laukams, apibrėžtiems ant sveikareikšmės gardelės $\mathbf{Z}^d$, $d \geqslant 1$, kai dėmenys turi baigtinius antrosios eilės momentus. Įvertis gautas Liapunovo trupmenos terminais, kuris nepriklausomu atveju ($m = 0$) yra tikslus.