LMD mokslo darbai, 24–28
1998, Vilnius

One formula for the fourth shifted moment of the weighted Riemann zeta-function

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1. While one considers mean values of Dirichlet series

$$\sum_{m=1}^{\infty} a_m m^{-s} \quad (\operatorname{Re} s > \sigma_0),$$

he definitely encounters with the evaluation of sums of the coeficients a_m or their forms. The asymptotic formula for the fourth moment of the Riemann zeta-function

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = T\, P_4(\log T) + E_2(T)$$

was obtained in this way, here $P_4(x)$ means a polynomial of the fourth degree of x. The estimates of the error term, first $E_2(T) = O\left(T^{7/8+\varepsilon}\right)$ by D.R. Heath-Brown [1], and the lattest $E_2(T) = O\left(T^{2/3}\log^e T\right)$ by A. Ivić and Y. Motohashi [2] were deduced on considering the sum

$$\sum_{n \leqslant x} d(n)\, d(n+r).$$

The aim of this paper is to obtain the formula expressing the fourth shifted moment of the weighted Riemann zeta-function as the integral of expressions of the coeficients $\sigma_{-a}(n)$. To derive such result we will use the classical methods of analytic number theory, applying the innovations introduced by M. Jutila [3]. Let

$$\delta = \frac{1}{T}, \quad T^{-1+\varepsilon} \ll U \ll T^{-\frac{1}{2}}, \quad U_1 = U \log T, \quad \Delta = U^{-1}\log T$$

and define

$$\varphi(z) = \sum_{n=1}^{\infty} \sigma_a(n) e^{-nz} - \frac{1}{z}\zeta(1+a) - \Gamma(1-a)\zeta(1-a) z^{a-1}.$$

LEMMA. *Let T be a large positive number tending to infinity, and let*

$$\omega(t) = \pi^{-\frac{1}{2}} U \int_{T_1-t}^{T_2-t} \exp\left(2\delta v - v^2 U^{2}\right) dv.$$

Then

$$\int_{T_1-\Delta}^{T_2+\Delta} \omega(t)\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left|\zeta\left(\frac{1}{2}+a+it\right)\right|^2 dt$$

$$= 2\int_{T_1}^{T_2} e^{2\delta\tau} \int_{-U_1}^{U_1}\int_0^{\infty} \varphi\left(2\pi i x e^{u-i\delta}\right)\varphi\left(-2\pi i x e^{-u+i\delta}\right) e^{2\tau u i - u^2/U^2} dx\, du\, d\tau + O(TU).$$

Notation: $s = \sigma + it$ is a complex variable, $a, c_1, c_2, \ldots$ are absolute positive constants (not necessarily the same at each occurence); Vinogradov symbol $f(x) \ll g(x)$ means that $|f(x)| \leqslant Cg(x)$ for $x \geqslant x_0$, with absolute constant C and a positive function $g(x)$; $f(x) = O(g(x))$ means the same as $f(x) \ll g(x)$; also the symbol "$\asymp$" implies "$\ll$" and "$\gg$"; we write ε for an arbitrarily small positive number, not necessarily the same at each occurence and, as usually, we note $\sigma_a(n) = \sum_{d|n} d^a$.

2. We start the proof of the lemma from the well-known formula (see [4])

$$\zeta(s)\,\zeta(s+a) = \sum_{n=1}^{\infty} \frac{\sigma_{-a}(n)}{n^s}, \qquad \left(\sigma > \max\{1, \operatorname{Re}(-a)+1\}\right).$$

Assuming $\operatorname{Re} z > 0$, we have

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\zeta(s)\zeta(s+a)\, z^{-s} ds = \sum_{n=1}^{\infty} \frac{\sigma_{-a}(n)}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)(nz)^{-s} ds$$

$$= \sum_{n=1}^{\infty} \sigma_{-a}(n) e^{-nz}$$

the interchange of integration and summation was possible for the absolute convergence of the integral. We move now the line of integration to $\sigma = \alpha$, where $0 < \alpha < \min\{1, \operatorname{Re}(-a)+1\}$. The integrated function has two poles at the points $s = 1$ and $s = 1 - a$, we sign their sum as

$$R(z) = \frac{1}{z}\zeta(1+a) + \Gamma(1-a)\zeta(1-a) z^{a-1}. \tag{1}$$

We define the function

$$\varphi(z) = \sum_{n=1}^{\infty} \sigma_a(n) e^{-nz} - R(z). \tag{2}$$

Hence,

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\zeta(s)\zeta(s+a)z^{-s}ds=\varphi(z). \tag{3}$$

Assuming $z=ixe^{-i\delta}$ with $0<\delta<\frac{\pi}{2}$, it is easy to prove that the functions

$$\varphi(ixe^{-i\delta})\qquad\text{and}\quad \Gamma(s)\zeta(s)\zeta(s+a)e^{-i\left(\frac{\pi}{2}-\delta\right)s}$$

are forming the Mellin's pair. Whence, the Parcevall's identity implies the equality

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Gamma(\sigma+it)\zeta(\sigma+it)\zeta(\sigma+a+it)\right|^2 e^{(\pi-2\delta)t}dt\\ =\int_0^{\infty}\left|\varphi(ixe^{-i\delta})\right|^2\cdot x^{2\sigma-1}dx\,.\end{aligned} \tag{4}$$

Further we assume that $\sigma=\frac{1}{2}$. Since

$$\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2=\frac{\pi}{\cosh\pi t},$$

we obtain that

$$\int_{-\infty}^{\infty}\frac{e^{\pi t}}{2\cosh\pi t}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left|\zeta\left(\frac{1}{2}+a+it\right)\right|^2e^{-2\delta t}dt=\int_0^{\infty}\left|\varphi\left(ixe^{-i\delta}\right)\right|^2dx. \tag{5}$$

This identity is valid for real positive parameter δ. Modifying the right-hand side of the equality (5), it is easy to show that the validity of (5) may be analytically continuated to the strip $0<\operatorname{Re}\delta<\pi$. Namely, writing $\xi=\delta+iu$, we have that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{\pi t}}{2\cosh\pi t}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left|\zeta\left(\frac{1}{2}+a+it\right)\right|^2e^{-2\xi t}dt\\ =2\pi\int_0^{\infty}\varphi\left(2\pi ixe^{-i\xi}\right)\cdot\varphi\left(-2\pi ixe^{i\xi}\right)dx,\end{aligned} \tag{6}$$

for $0<\operatorname{Re}\xi<\pi$.

The key argument of the method is to assume u lying in a certain short real interval around zero and to average the equality (6). Let T be a large positive number, and let

$$\delta=\frac{1}{T},\quad T^{-1+\varepsilon}\ll U\ll T^{\frac{-1}{2}},\quad U_1=U\log T,\quad \Delta=U^{-1}\log T. \tag{7}$$

We multiply the both sides of (6) by the factor

$$\pi^{-1}e^{2\tau\xi-u^2/U^2} \quad \text{for } \tau \asymp T,$$

and integrate over $|u| \leqslant U_1$ and $T_1 \leqslant \tau \leqslant T_2$ for some $T_1, T_2 \asymp T$. Introducing the notation $I(T_1, T_2)$ for final integral, we obtain that

$$I(T_1, T_2) = \int_{T_1}^{T_2} I(\tau)\, d\tau, \tag{8}$$

where

$$I(\tau) = \pi^{-1}e^{2\delta\tau} \int_{-\infty}^{\infty} \frac{e^{\pi t}}{2\cosh \pi t} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left|\zeta\left(\frac{1}{2} + a + it\right)\right|^2 e^{-2\delta t} \times \int_{-U_1}^{U_1} e^{2(\tau-t)ui-u^2/U^2} du\, dt, \tag{9}$$

and, on the other hand,

$$I(T_1, T_2) = 2\int_{T_1}^{T_2} e^{2\delta\tau} \int_{-U_1}^{U_1} \int_0^{\infty} \varphi\left(2\pi i x e^{u-i\delta}\right)\varphi\left(-2\pi i x e^{-u+i\delta}\right)e^{2\tau ui-\frac{u^2}{U^2}} dx\, du\, d\tau. \tag{10}$$

To prove the lemma we evaluate the two expressions of the integral $I(T_1, T_2)$, separately. Let start from the integral (9). We will need the following known formula

$$\int_{-\infty}^{\infty} e^{Ax-Bx^2} dx = \sqrt{\frac{\pi}{B}}\, e^{A^2/4B} \quad \text{for } \operatorname{Re} B > 0. \tag{11}$$

First we simplify the integral (9). It is easy to see that we may restrict the range of integration over t to $[0, +\infty)$ within the accuracy of the error term $O(U)$. For such t the factor in the integral $e^{\pi t}/2\cosh \pi t$ may be simplified to 1, the error term of this step is also $O(U)$. Finally, to use the formula (11), the integration over U may be extended to the whole real line. Hence,

$$I(T_1, T_2) = \int_0^{\infty} \omega(t) \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left|\zeta\left(\frac{1}{2} + a + it\right)\right|^2 dt + O(TU), \tag{12}$$

where

$$\omega(t) = \pi^{-\frac{1}{2}} U \int_{T_1-t}^{T_2-t} \exp(2\delta v - v^2 U^2) dv,$$

by (11), after the additional substitution. From the definition of the function $\omega(t)$, it is easy to deduce that $\omega(t) \ll \exp(-\log^2 T)$ if $t \notin [T_1 - \Delta, T_2 + \Delta]$. This estimate implies

$$I(T_1, T_2) = \int_{T_1-\Delta}^{T_2+\Delta} \omega(t) \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left|\zeta\left(\frac{1}{2} + a + it\right)\right|^2 dt + O(TU). \tag{13}$$

Note that

$$\omega(t) = 1 + O((TU)^{-2})$$

in the range $[T_1 + \Delta, T_2 - \Delta]$. Therefore the integral $I(T_1, T_2)$ is close to the fourth shifted moment of the Riemann zeta-function. Whence, summing the estimates (8)–(13) we obtain the desired lemma.

Applying deep spectral methods to the integral (10), one could deduce the asymptotic formula for the fourth shifted power moment of the Riemann zeta-function with an estimate for the error term.

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Viena paslinktojo ketvirto Rymano dzeta funkcijos su svoriu vidurkio išraiška

A. Kačėnas

Šiame straipsnyje nagrinėjamas Rymano dzeta funkcijos paslinktasis ketvirtasis vidurkis su svoriu. Naudojantis klasikiniais analizinės skaičių teorijos metodais, gaunama jo tiksli išraiška per koeficientų $\sigma_a(n)$ funkcijų integralus. Ši išraiška gali būti naudojama, įrodant Rymano dzeta funkcijos pasliktojo ketvirtojo vidurkio asimptotinę formulę.