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On the universality of Dirichlet series of holomorphic cusp forms

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1. Introduction

Let $F(z)$ be a holomorphic cusp form of weight κ for the full modular group $SL(2, \mathbb{Z})$. Assume that $F(z)$ is a normalized eigenform. Then $F(z)$ has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}, \qquad c(1) = 1.$$

Let $s = \sigma + it$ be a complex variable. Consider the Dirichlet series

$$\varphi(s) = \varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

E.Hecke proved [2] that this series absolutely converges for $\sigma > (\kappa + 1)/2$, and can be continued analytically to an entire function.

Let $D = \{s \in \mathbb{C} : \ \kappa/2 < \sigma < (\kappa + 1)/2\}$, where $\mathbb{C}$ denotes the complex plane. The purpose of this paper is to prove the following universality theorem for the function $\varphi(s)$.

We use the notation

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T], \ldots\}$$

for $T > 0$, where in place of dots we write a condition satisfied by τ, and $\operatorname{meas}\{A\}$ denotes the Lebesgue measure of the set A.

THEOREM. *Let K be a compact subset of D with connected complement, and let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$*

$$\liminf_{T \to \infty} \nu_T\left(\sup_{s \in K} \left| \varphi(s + i\tau) - f(s) \right| < \varepsilon \right) > 0.$$

This theorem is proved in a preprint of the second and the third authors, and the proof is based on the idea of [3] with some new ideas of the third author.

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Let $c_p = c(p)p^{(1-\kappa)/2}$. Note that in [3] the theorem was proved under the assumption of the existence of $\eta > 0$ such that

$$\sum_{\substack{p \\ |c_p|<\eta}} \frac{1}{p^\delta} < \infty \tag{1}$$

for $\delta > 1/2$. Now, the theorem assures the universality of the function $\varphi(s)$ unconditionally.

The method of the proof of the theorem is the same as in [3], but some new arguments to obtain the denseness of one set of convergent series are used. This allows to remove the condition (1).

2. A limit theorem for the function $\varphi(s)$

The function $\varphi(s)$ for $\sigma > (\kappa+1)/2$ has the Euler product expansion

$$\varphi(s) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $c(p) = \alpha(p) + \beta(p)$.

Let $N > 0$, $D_N = \{s \in \mathbb{C} : \kappa/2 < \sigma < (\kappa+1)/2, |t| < N\}$. Denote by $H(D_N)$ the space of analytic on D_N functions equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Define on $\big(H(D_N), \mathcal{B}(H(D_N))\big)$ the probability measure

$$P_T(A) = \nu_T\big(\varphi(s + i\tau) \in A\big).$$

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. The infinitedimensional torus Ω is a compact topological Abelian group. Denote by m_H the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Thus we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p. Define the $H(D_N)$-valued random element $\varphi(s, \omega)$ on $(\Omega, \mathcal{B}(\Omega), m_H)$ by the formula

$$\varphi(s, \omega) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}$$

for $s \in D_N$. Denote by P_φ the distribution of the random element $\varphi(s, \omega)$.

LEMMA 1. *The probability measure P_T converges weakly to P_φ as $T \to \infty$.*

Proof. In [3] the lemma was proved on the space $H(D)$, where $D = \{s \in \mathbb{C} : \sigma > \kappa/2\}$. Clearly, from this the lemma follows immediately.

3. A denseness lemma

Let, for $|z| < 1$,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots .$$

Define

$$f_p(s) = f_p(s; a_p) = -\log\left(1 - \frac{\alpha(p)a_p}{p^s}\right) - \log\left(1 - \frac{\beta(p)a_p}{p^s}\right)$$

for $s \in D_N$ and $a_p \in \gamma$.

LEMMA 2. *The set of all convergent series*

$$\sum_p f_p(s; a_p)$$

is dense in $H(D_N)$.

Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_N, $D_{1,N} = \{s \in \mathbb{C} : 1/2 < \delta < 1, |t| < N\}$, and let $h(s) = s - (\kappa - 1)/2$. Then

$$\mu h^{-1}(A) = \mu(h^{-1}A), \qquad A \in \mathcal{B}(\mathbb{C}),$$

is a complex measure with compact support contained in $D_{1,N}$. Define

$$\varrho(z) = \int_{\mathbb{C}} e^{-sz}\, d\mu h^{-1}(s), \qquad z \in \mathbb{C}.$$

LEMMA 3. *Suppose that*

$$\sum_p |c_p|\, |\varrho(\log p)| < \infty.$$

Then $\varrho(z) \equiv 0$.

Proof of the theorem is based on the following variant of the Bernstein theorem.

LEMMA 4. *Let $f(s)$ be an entire function of exponential type, and let $\{\lambda_m\}$ be a sequence of complex numbers. Let α, β and δ be positive real numbers such that*

1^0 $\limsup\limits_{y\to\infty} \dfrac{\log|f(\pm iy)|}{y} \leqslant \alpha;$

2^0 $|\lambda_m - \lambda_n| \geqslant \delta|m-n|;$

3^0 $\lim\limits_{m\to\infty} \dfrac{\lambda_m}{m} = \beta;$

4^0 $\alpha\beta < \pi.$

Then

$$\limsup_{m\to\infty} \frac{\log|f(\lambda_m)|}{|\lambda_m|} = \limsup_{r\to\infty} \frac{\log|f(r)|}{r}.$$

Proof is given in [4].

We also need the following lemma.

LEMMA 5. *Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the half-plane $\sigma > \sigma_0$, and let*

$$f(z) = \int_{\mathbb{C}} e^{sz}\, d\mu(s)\,.$$

If $f(z) \not\equiv 0$, then

$$\limsup_{r\to\infty} \frac{\log|f(r)|}{r} > \sigma_0.$$

Proof of the lemma can be found in [4].

Proof of Lemma 3. We apply Lemma 4 with $f = \varrho$. Since the support of the measure μh^{-1} is included in $D_{1,N}$, we obtain that

$$\big|\varrho(\pm iy)\big| \leqslant e^{Ny} \int_{\mathbb{C}} \big|d\mu h^{-1}(s)\big|$$

for $y > 0$. Therefore we can take $\alpha = N$ in the condition 1^0 of Lemma 4. Let a fixed positive number β satisfy $\beta < \pi/N$. Consider the set A of all positive integers m such that there exists a real number $r \in \big((m-1/4)\beta, (m+1/4)\beta\big)$ with $|\varrho(r)| \leqslant e^{-r}$.

We fix a number μ, satisfying $0 < \mu < 1$, and put $\mathcal{P}_\mu = \{p \text{ is primes}, \ |c_p| > \mu\}$. Then the condition of the lemma implies

$$\sum_{p\in\mathcal{P}_\mu} |\varrho(\log p)| < \infty. \tag{2}$$

On the other hand, we have

$$\sum_{p\in\mathcal{P}_\mu} |\varrho(\log p)| \geqslant \sum_{m\notin A} {\sum_m}' |\varrho(\log p)| \geqslant \sum_{m\notin A} {\sum_m}' \frac{1}{p}, \tag{3}$$

where ${\sum_m}'$ denotes the sum running over all primes $p \in \mathcal{P}_\mu$ satisfying $\log a \leqslant \log p \leqslant \log b$ with $a = \exp\{(m-1/4)\beta\}$, $b = \exp\{(m+1/4)\beta\}$. Thus (2) and (3) yield

$$\sum_{m\notin A} \sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant b}} \frac{1}{p} < \infty. \tag{4}$$

Let $\pi_\mu(x)$ be the number of primes $p \in \mathcal{P}_\mu$ up to x. It is known [1] that $|c_p| \leqslant 2$. Therefore, for $a \leqslant u \leqslant b$,

$$\begin{aligned} \sum_{a<p\leqslant u} c_p^2 &\leqslant 4 \sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant u}} 1 + \mu^2 \sum_{\substack{p\notin\mathcal{P}_\mu \\ a<p\leqslant u}} 1 \\ &= (4-\mu^2)\big(\pi_\mu(u) - \pi_\mu(a)\big) + \mu^2\big(\pi(u) - \pi(a)\big). \end{aligned} \tag{5}$$

On the other hand, by Rankin's formula [5]

$$\sum_{p\leqslant x} c_p^2 = \pi(x)\big(1+o(1)\big), \qquad x\to\infty,$$

we have

$$\sum_{a<p\leqslant u} c_p^2 = \pi(u)\big(1+o(1)\big) - \pi(a)\big(1+o(1)\big), \quad m\to\infty. \tag{6}$$

We fix a positive parameter δ satisfying $1+\delta < e^{\beta/2}$, and let $0<\varepsilon<\delta/100$. If $m \geqslant m_0(\varepsilon)$, then, for any $u \geqslant a(1+\delta)$,

$$\pi(u)\big(1+o(1)\big) \geqslant \pi(u)(1-\varepsilon),$$
$$\pi(a)\big(1+o(1)\big) \leqslant \pi(a)(1+\varepsilon).$$

Hence

$$\pi(u)\big(1+o(1)\big) - \pi(a)\big(1+o(1)\big) \geqslant \big(\pi(u)-\pi(a)\big) - \varepsilon\big(\pi(u)+\pi(a)\big). \tag{7}$$

Since $u \geqslant a(1+\delta)$, we easily find, for $m \geqslant m_0(\varepsilon)$

$$\pi(u)-\pi(a) \geqslant \frac{u}{\log u}(1-\varepsilon) - \frac{a}{\log a}(1+\varepsilon) \geqslant \frac{a}{\log a}\frac{\delta}{2}. \tag{8}$$

On the other hand, if $u \leqslant b = Ba$, $B = e^{\beta/2}$, then

$$\pi(u)+\pi(a) \leqslant \pi(b)+\pi(a) \leqslant \frac{a}{\log a}(2B+2).$$

Therefore this and (8) yield

$$\pi(u)+\pi(a) \leqslant \frac{4B+2}{\delta}\big(\pi(u)-\pi(a)\big).$$

Hence and from (6) we find

$$\pi(u)\big(1+o(1)\big) - \pi(a)\big(1+o(1)\big) \geqslant \big(\pi(u)-\pi(a)\big)(1+o(1)), \qquad m\to\infty.$$

Thus, in view of (5)

$$\pi_\mu(u)-\pi_\mu(a) \geqslant \frac{1-\mu^2}{4-\mu^2}\big(\pi(u)-\pi(a)\big)(1+o(1)), \qquad m\to\infty,$$

for $u \geqslant a(1+\delta)$. Therefore, using partial summation,

$$\begin{aligned}\sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\leqslant b}} \frac{1}{p} &\geqslant \big(\pi_\mu(b)-\pi_\mu(a)\big)\frac{1}{b} + \int\limits_{a(1+\delta)}^{b} \big(\pi_\mu(u)-\pi_\mu(a)\big)\frac{du}{u^2}\\ &\geqslant \frac{1-\mu^2}{4-\mu^2}\Bigg(\sum_{a(1+\delta)<p\leqslant b}\frac{1}{p}\Bigg)\big(1+o(1)\big), \qquad m\to\infty.\end{aligned} \tag{9}$$

Since

$$\sum_{a(1+\delta)<p\leqslant b} \frac{1}{p} = \left(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\right)\frac{1}{m} + O\left(\frac{1}{m^2}\right),$$

we find from (9)

$$\sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant b}} \frac{1}{p} \geqslant \frac{1-\mu^2}{4-\mu^2}\left(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\right)\frac{1}{m}(1+o(1)) + O\left(\frac{1}{m^2}\right). \tag{10}$$

Since $0 < \mu < 1$ and $1+\delta < e^{\beta/2}$, we see that

$$\frac{1-\mu^2}{4-\mu^2}\left(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\right) > 0.$$

Therefore, from (4) and (10) we obtain

$$\sum_{m\notin A} \frac{1}{m} < \infty.$$

Now, we derive from Lemma 4 that

$$\limsup_{r\to\infty} \frac{\log|\varrho(r)|}{r} \leqslant -1. \tag{11}$$

But if $\varrho(z) \not\equiv 0$, then Lemma 5 gives

$$\limsup_{r\to\infty} \frac{\log|\varrho(r)|}{r} > -1,$$

which contradicts with (11). Therefore $\varrho(z) \equiv 0$.

Now using Lemma 3 the proof of Lemma 2 runs in the same way as in [3].

The proof of the theorem uses Lemma 1 and Lemma 2 and is similar to that from [3].

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holomorfinių parabolinių formų Dirichlet eilučių universalumą

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Straipsnyje įrodyta universalumo teorema apie analizinės funkcijos tolygią aproksimaciją Dirichlet eilutės, susietos su holomorfine paraboline forma, postūmiais.