On estimation of the number of zeros of linear combinations of certain zeta-functions

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Let $s = \sigma + it$ be a complex variable. In this note we consider zeta-functions $\varphi_l(s)$, introduced by K. Matsumoto in [3]. Let

$$A_{lm}(x) = \prod_{j=1}^{g_l(m)} \left(1 - a_{lm}^{(j)} x^{f_l(j,m)}\right),$$

where $g_l(m)$, $f_l(j,m) \in \mathbb{N}$ and $a_{lm}^{(j)} \in \mathbb{C}$. Here $\mathbb{N}$ and $\mathbb{C}$, as usual, denote the sets of all natural and complex numbers, respectively. Then the functions $\varphi_l(s)$ are defined by

$$\varphi_l(s) = \prod_{m=1}^{\infty} A_{lm}^{-1}(p_m^{-s}) \tag{1}$$

where p_m stands for the mth prime number. If

$$g_l(m) \leqslant c_{1l} p_m^{\alpha_l}, \qquad |a_{lm}^{(j)}| \leqslant p_m^{\beta_l}$$

with positive constants c_{1l} and non-negative constants α_l and β_l, then the infinite product in (1) converges absolutely for $\sigma > \alpha_l + \beta_l + 1$ and defines a holomorphic function with no zeros.

Now let $r \geqslant 2$, $u_l \in \mathbb{C}$ and

$$Z(s) = \sum_{l=1}^{r} u_l \varphi_l(s).$$

Suppose that at least two of numbers u_l are distinct from zero. Let a real number ϱ_0 be defined by $\max(\alpha_l + \beta_l + 1/2) \leqslant \varrho_0 < \min(\alpha_l + \beta_l + 1)$. Assume that the functions $\varphi_l(s)$ are analytically continuable to the strip $D = \{s \in \mathbb{C} : \varrho_0 < \sigma < \min(\alpha_l + \beta_l + 1)\}$, and that in the half-plane $\sigma > \varrho_0$ the following estimates are satisfied

$$\varphi_l(\sigma + it) = B|t|^{c_1}, \qquad c_1 > 0,$$

$$\int_0^T |\varphi_l(\sigma + it)|^2 \, dt = BT, \qquad T \to \infty.$$

Here B denotes a number (not always the same) bounded by a constant.

Let

$$B_l(m) = \frac{1}{p_m^{\alpha+\beta}} \sum_{\substack{j=1 \\ f_l(j,m)=1}}^{g_l(m)} a_{lm}^{(j)}.$$

In [2] we assumed that $B_l(m)$ is constant for $m \in M_j = \{m : \; p_m \in P_j\}$, where P_j, $j = 1, \ldots, k \geqslant r$, are sets of prime numbers such that $P_j \bigcap P_l = \varnothing$ for $j \neq l$, $P = \bigcup_{j=1}^{k} P_j,$, ($P$ denotes the set of all primes) and

$$\sum_{\substack{p \leqslant x \\ p \in P_j}} = b_j \log \log x + d_j + \varrho_j(x),$$

where $b_1 + \ldots + b_k = 1$, $b_j > 0$, $\varrho_j(x) = B(\log x)^{-\theta_j}$ with $\theta_j > 1$, and d_j are some real numbers. In this note we replace the last condition by the following: the sets P_j have a positive density.

Let, for $m \in M_j$,

$$B_1(m) = B_{1j},$$
$$\ldots\ldots\ldots\ldots$$
$$B_r(m) = B_{rj},$$

and let

$$G_{kr} = \begin{pmatrix} B_{11} & \ldots & B_{1r} \\ \ldots & \ldots & \ldots \\ B_{k1} & \ldots & B_{kr} \end{pmatrix}.$$

THEOREM. *Let all above conditions be satisfied. Suppose that rank $(G_{kr}) = r$. Then for all σ_1, σ_2, $\varrho < \sigma_1 < \sigma_2 < \min(\alpha_l + \beta_l + 1)$, there exists a constant $c = c(\sigma_1, \sigma_2) > 0$ such that, for sufficiently large T, the function $Z(s)$ has more than cT zeros lying in the region $\sigma_1 < \sigma < \sigma_2, 0 \leqslant t \leqslant T$.*

Proof of the theorem coinsides with that of Theorem 1 from [2], however, it uses one new lemma on entire functions of exponential type obtained by the second author. The theorem is stated by the first author.

Let $\mathcal{P}$ be a set of prime numbers having a positive density, i.e.,

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \#\{p \in \mathcal{P} : \; p \leqslant x\} = d > 0, \tag{2}$$

where, as usual,

$$\pi(x) = \sum_{p \leqslant x} 1.$$

LEMMA. *Let $f(s)$ be an entire function of exponential type, and let*

$$\limsup_{r \to \infty} \frac{\log |f(r)|}{r} > -1.$$

Then

$$\sum_{p \in \mathcal{P}} |f(\log p)| = \infty.$$

Proof. Let $\alpha > 0$ be such that

$$\limsup_{y \to \infty} \frac{\log |f(\pm iy)|}{y} \leqslant \alpha.$$

Let us fix a positive number β such that $\alpha\beta < \pi$. Suppose, on the contrary, that

$$\sum_{p \in \mathcal{P}} |f(\log p)| < \infty. \tag{3}$$

Consider the set $A = \{m \in \mathbb{N} : \exists r \in ((m - 1/4)\beta, (m + 1/4)\beta)$ and $|f(r)| \leqslant e^{-r}\}$. Let, for brevity,

$$\pi_{\mathcal{P}}(x) = \sum_{\substack{p \leqslant x \\ p \in \mathcal{P}}} 1.$$

We have that

$$\sum_{p \in \mathcal{P}} |f(\log p)| \geqslant \sum_{m \notin A} \sideset{}{'}\sum_{m} |f(\log p)| \geqslant \sum_{m \notin A} \sideset{}{'}\sum_{m} \frac{1}{p}, \tag{4}$$

where $\sideset{}{'}\sum_{m}$ denotes the sum running over all prime numbers $p \in \mathcal{P}$ and satisfying $(m - 1/4)\beta < \log p < (m + 1/4)\beta$. Therefore, putting

$$a = \exp\left\{\left(m - \frac{1}{4}\right)\beta\right\}, \qquad b = \exp\left\{\left(m + \frac{1}{4}\right)\beta\right\},$$

we can write

$$\sideset{}{'}\sum_{m} \frac{1}{p} = \sum_{\substack{p \in \mathcal{P} \\ a < p < b}} \frac{1}{p}.$$

By partial summation

$$\sum_{\substack{p \in \mathcal{P} \\ a < p < b}} \frac{1}{p} = \frac{1}{b} \sum_{\substack{p \in \mathcal{P} \\ a < p < b}} 1 + \int_a^b \left(\sum_{\substack{p \in \mathcal{P} \\ a < p < u}} 1\right) \frac{du}{u^2}. \tag{5}$$

Clearly,

$$\sum_{\substack{p \in \mathcal{P} \\ a < p < u}} 1 = \pi_{\mathcal{P}}(u) - \pi_{\mathcal{P}}(a),$$ (6)

and, by (2),

$$\pi_{\mathcal{P}}(x) = d\pi(x)(1 + o(1)), \qquad x \to \infty.$$

Therefore, for any $\varepsilon > 0$, there exists a number $x_0 = x_0(\varepsilon)$ such that

$$\pi_{\mathcal{P}}(u) \geqslant d\pi(u)(1 - \varepsilon),$$
$$\pi_{\mathcal{P}}(a) \leqslant d\pi(a)(1 + \varepsilon)$$

if $x \geqslant x_0$. Hence and from (6)

$$\sum_{\substack{p \in \mathcal{P} \\ a < p < u}} 1 \geqslant d\big((\pi(u) - \pi(a)) - \varepsilon(\pi(a) + \pi(u))\big).$$ (7)

If x_0 is large enough, then, for $a \geqslant x_0$,

$$\pi(u) - \pi(a) \geqslant \frac{u}{\log u} - \frac{a}{\log a} - \varepsilon\left(\frac{u}{\log u} + \frac{a}{\log a}\right),$$

$$\pi(u) + \pi(a) \leqslant \frac{u}{\log u}(1 + \varepsilon) + \frac{a}{\log a}(1 + \varepsilon).$$

Hence

$$\frac{1}{2}\big(\pi(u) - \pi(a)\big) - \varepsilon\big(\pi(u) + \pi(a)\big) \geqslant \frac{u}{2\log u} - \frac{a}{2\log a} - \varepsilon\left(\frac{3}{2} + \varepsilon\right)\left(\frac{u}{\log u} + \frac{a}{\log a}\right). \quad (8)$$

Obviously, $\frac{b}{a} = \exp\{\beta/2\}$. Let η be a positive number satisfying $1 < \eta < \exp\{\beta/2\}$, and consider the case $u \geqslant \eta a$. Then

$$\frac{a}{\log a} \leqslant \frac{1 + \varepsilon}{\eta} \frac{u}{\log u}$$

if $u \geqslant x_0$ is sufficiently large. Therefore, by (8),

$$\frac{1}{2}\big(\pi(u) - \pi(a)\big) - \varepsilon\big(\pi(u) + \pi(a)\big) \geqslant \left(\frac{1}{2} - \frac{1 + \varepsilon}{2\eta} - 2\varepsilon\left(\frac{3}{2} + \varepsilon\right)\right)\frac{u}{\log u} > 0$$

if we choose ε sufficiently small. Hence from (7) we obtain

$$\sum_{\substack{p \in \mathcal{P} \\ a < p < u}} 1 \geqslant \frac{d}{2}\big(\pi(u) - \pi(a)\big), \qquad u \geqslant \eta a.$$

Therefore, from (5) we get by partial summation

$$\sum_{\substack{p \in \mathcal{P} \\ a < p < b}} \frac{1}{p} \geq \frac{d}{2b} \big(\pi(b) - \pi(a) \big) + \frac{d}{2} \int_{\eta a}^{b} \big(\pi(u) - \pi(a) \big) \frac{du}{u^2} \geq \frac{d}{2} \sum_{\eta a < p < b} \frac{1}{p}. \tag{9}$$

It is not difficult to see that

$$\sum_{\eta a < p < b} \frac{1}{p} = \log \log b - \log \log(\eta a) + B \exp \big\{ - c_2 \sqrt{\log(\eta a)} \big\} = \left(\frac{1}{2} - \frac{\log \eta}{\beta} \right) \frac{1}{m} + \frac{B}{m^2}.$$

Here β is a fixed positive number. Therefore, if we choose η sufficiently close to 1, we have

$$\left(\frac{1}{2} - \frac{\log \eta}{\beta} \right) \stackrel{def}{=} c_3 > 0.$$

Now (9) yields

$$\sum_{\substack{p \in \mathcal{P} \\ a < p < b}} \frac{1}{p} \geq \frac{d}{2} \left(\frac{c_3}{m} + \frac{B}{m^2} \right) = \frac{c_3 d}{2} \frac{1}{m} + \frac{B}{m^2}.$$

This and (4) imply

$$\sum_{m \notin A} \left(\frac{c_3 d}{2} \frac{1}{m} + \frac{B}{m^2} \right) \leq \sum_{p \in \mathcal{P}} |f(\log p)| < \infty,$$

hence

$$\sum_{m \notin A} \frac{1}{m} < \infty. \tag{10}$$

Let $A = \{a_m : a_1 < a_2 < \ldots\}$. Then, by (10), we obtain that

$$\lim_{m \to \infty} \frac{a_m}{m} = 1. \tag{11}$$

By the definition of the set A, there exists a sequence $\{\lambda_m\}$ such that

$$\left(a_m - \frac{1}{4} \right) \beta \leq \lambda_m \leq \left(a_m + \frac{1}{4} \right) \beta,$$

and

$$|f(\lambda_m)| \leq e^{-\lambda_m}.$$

Hence, in view of (11),

$$\lim_{m \to \infty} \frac{\lambda_m}{m} = \beta,$$

and

$$\limsup_{m \to \infty} \frac{\log |f(\lambda_m)|}{\lambda_m} \leq -1.$$

Applying a version of the Bernstein theorem (Theorem 6.4.12 of [1]), we find that

$$\limsup_{r \to \infty} \frac{\log |f(r)|}{r} \leq -1.$$

This contradicts to the assumption of the lemma. Hence the assertion (3) is false, and the lemma is proved.

REFERENCES

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Apie kai kurių dzeta funkcijų tiesinių kombinacijų nulių skaičiaus įvertį

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Straipsnyje nagrinėjamos dzeta funkcijos, apibrėžiamos polinomine Eulerio sandauga, ir gaunamas šių funkcijų tiesinių kombinacijų nulių skaičiaus įvertis iš apačios.