LMD mokslo darbai, 76–79
1998, Vilnius

On the divisor sums in arithmetical progressions

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Let $\sigma(n)$ denote the sum of the divisors of the positive integer n and let $D(x)$ be the number of positive integers n for which $\sigma(n) \leqslant x$, i.e.

$$D(x) = \sum_{\sigma(n)\leqslant x} 1.$$

In the papers [1]–[4] the asymptotic behaviour of the sum $D(x)$ as $x \to \infty$ was considered. In [3] and [2] the existence of the limit

$$\lim_{x\to\infty} \frac{D(x)}{x} = d,$$

where

$$d = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{j=1}^{\infty} \frac{1}{1 + p + \cdots + p^j}\right),$$

is proved. P. T. Bateman [1] has proved the equality

$$D(x) = dx + O_\varepsilon\left(x \exp\left\{-\left(1 - \varepsilon\right)\sqrt{\frac{1}{2}\log x \log\log x}\right\}\right) \tag{1}$$

for any positive number ε. In [4] for the remainder term of equality (1), the estimation

$$O\left(x \exp\left\{-\left(1 + \frac{c_1 \log\log\log x}{\log\log x}\right)\sqrt{\frac{1}{2}\log x \log\log x}\right\}\right), \quad c_1 > 0$$

is obtained.

In this paper the similar result for

$$D(x; k, l) := \sum_{\substack{\sigma(n)\leqslant x \\ n\equiv l \,(\mathrm{mod}\, k)}} 1; \quad (l, k) = 1, \ k \geqslant 2$$

is proved. Besides, the numerical results of frequencies

$$\nu_x(k, l) = \frac{1}{x} D(x; k, l)$$

for $k = 5$ are presented (see Table 1). The data indicate the stability of the frequencies as $x \to \infty$ with the same value for different l, $(l, 5) = 1$.

Table 1

x	500	1000	2000	5000	10000	20000
$\nu_x(5,0)$	0,114	0,114	0,115	0,115	0,115	0,115
$\nu_x(5,1)$	0,136	0,140	0,140	0,139	0,139	0,139
$\nu_x(5,2)$	0,140	0,141	0,137	0,139	0,138	0,139
$\nu_x(5,3)$	0,140	0,135	0,139	0,138	0,139	0,139
$\nu_x(5,4)$	0,140	0,135	0,140	0,138	0,139	0,139
$D(x)$	0,668	0,668	0,6725	0,671	0,6719	0,6727

THEOREM. *For any integers $k \geqslant 2$ and l, $(l, k) = 1$, the asymptotic equality*

$$\nu_x(k,l) = d_k + O\left(k \exp\left\{ -\left(1 + \frac{c \log\log\log x}{\log\log x}\right)\sqrt{\frac{1}{2}\log x \log\log x}\right\}\right),$$

where $c > 0$,

$$d_k = \frac{d}{k}\prod_{p|k}\left(1 + \frac{1}{p-1}\right)\left(1 + \sum_{j=1}^{\infty}\frac{1}{1+p+\cdots+p^j}\right)^{-1},$$

as $x \to \infty$, is true.

The proof of the theorem is analytic and similar to that of analogous result for the Euler's function (see [5]). In this paper we only give the main ideas of the proof.

We consider the function

$$F(s) = \sum_{\substack{\sigma(n)\leqslant x \\ n\equiv l (\bmod k)}} \sigma(n)^{-s}, \quad s = \sigma + it, \quad \sigma > 1.$$

The analytic properties of $F(s)$ may be deduced from the properties of the functions

$$\Phi(s,\chi) = \sum_{n=1}^{\infty}\chi(n)\sigma(n)^{-s},$$

since

$$F(s) = \frac{1}{\varphi(k)}\sum_{\chi}\overline{\chi}(l)\Phi(s,\chi).$$

Here $\chi = \chi(n)$ is a Dirichlet character mod k and $\varphi(k)$ is the Euler totient function.

For the function $\Phi(s,\chi)$ we have

$$\Phi(s,\chi) = \prod_p\left(1 + \sum_{j=1}^{\infty}\chi^j(p)(p^j + p^{j-1} + \cdots + p + 1)^{-s}\right) = L(s,\chi)\cdot G(s,\chi),$$

where

$$L(s,\chi)=\prod_p\left(1-\chi(p)p^{-s}\right)^{-1}$$

is a Diriclet L-function and

$$G(s,\chi)=\prod_p\Bigl(1-\sum_{j=1}^{\infty}\chi^j(p)\bigl(p^{-s}(1+p+\cdots+p^{j-1})^{-s}$$
$$-(1+p+\cdots+p^j)^{-s}\bigr)\Bigr)=:\prod_p\left(1-\omega_p(s)\right).$$

As in the Lemma 1 of the paper [4] for any χ (mod k) and for $\sigma>0$, we obtain

$$|\omega_p(s)|\leqslant c_2|s|p^{-1-\sigma}.$$

Hence the function $G(s,\chi)$ is analytic in the half-plane $\sigma>0$ for every χ(mod k). Consequently the function $\Phi(s,\chi)$ for $\chi\neq\chi_0$ is also analytic on the half-plane $\sigma>0$. For the principal character $\chi=\chi_0$ we have the analyticity of $\Phi(s,\chi_0)$ in the half-plane $\sigma>0$ except the point $s=1$. The residue at the simple pole $s=1$ is $\varphi(k)d_k$.

Let $\tau=k|t|$, τ_0 be a fixed number (sufficiently large). Denote

$$\eta(\tau)=\frac{\log\log\tau}{\log\tau},\quad \tau\geqslant\tau_0.$$

Similarly as in [5] we can deduce the estimates:

$$F(s)=O\Bigl(\exp\Bigl\{\frac{c_3\log\tau}{\log\log\tau}\Bigr\}\Bigr),\quad 1-\eta(\tau)\leqslant\sigma\leqslant1,\quad \tau\geqslant\tau_0;$$

$$F(s)=O\bigl((\log\tau)^{c_4}\bigr),\quad 1\leqslant\sigma\leqslant2,\quad \tau\geqslant2.$$

Here c_3 and c_4 are positive constants. To prove the theorem, the contour integration may be applied:

$$E(x):=\frac{1}{x}\int_1^x D(u;k,l)du=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}F(s)x^s\frac{ds}{s(s+1)},\quad b>1.$$

The integration over contour K,

$$K=\{s:s=a+i\tau\ \text{ for }\ |\tau|\leqslant\tau_0;\ s=1-\eta(\tau)+i\tau\ \text{ if }\ |\tau|>\tau_0\},$$

where $a=1-\eta(\tau_0)$, gives the assertion of the theorem for $E(x)$. Then by applying the standard arguments we deduce the equality of the theorem.

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Skaičių iš aritmetinės progresijos daliklių sumos

E. Stankus

Pažymėkime $\sigma(n)$ skaičiaus n daliklių sumą ir tegu

$$D(x;k,l) := \sum_{\substack{\sigma(n)\leqslant x\\ n\equiv l \pmod k}} 1; \quad (l,k)=1,\ k\geqslant 2.$$

Darbe įrodoma, jog bet kuriems sveikiesiems $k \geqslant 2$ ir l, $(l,k)=1$, kai $x\to\infty$, galioja asimptotinė lygybė

$$D(x;k,l) = d_k x + O\left(xk\exp\left\{-\left(1+\frac{c\log\log\log x}{\log\log x}\right)\sqrt{\frac{1}{2}\log x\log\log x}\right\}\right),$$

kurioje c – teigiama konstanta,

$$d_k = \frac{1}{k}\prod_{p\nmid k}\left(1-\frac{1}{p}\right)\left(1+\sum_{j=1}^{\infty}\frac{1}{1+p+\cdots+p^j}\right).$$