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On the structure of tangent bundle of (*f*, *g*)-manifold defined by 2-forms

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In works of Russian and Japanese mathematicians the great attention is paid to the theory of structures in differentiable manifolds as well as to the theory of tangent bundle of these manifolds [2, 3].

We shall consider so-called (*f*, *g*)-structures and (*f*, *g*)-manifolds [1], which are some generalization of almost Hermitian structure and almost Hermitian manifolds. We shall analyse such (*f*, *g*)-structures, which are analogous to Kahlerian srtucture. It is shown in the note that some 2-forms, defined by structural covariant tensor of such a (*f*, *g*)-manifold M_n, induce in tangent bundle $T(M_n)$ the same structure as in the basic manifold.

1. Let M_n be n-dimensional differentiable manifold, x^i – local coordinates at point $M \in M_n$, $i, j, k, \ldots = 1, \ldots n$. Let us have in the M_n an affinor $f_i^{\ j}$ and a metric g_{ij} (not necessary positive definite), satisfying the conditions:

$$f_i^{\ j} f_j^{\ k} = \varpi \delta_i^k, \varpi = \pm 1, \tag{1}$$

$$f_i^{\ j} g_{jk} = f_{ik} = \rho f_{ki}, \rho = \pm 1. \tag{2}$$

Then M_n is called (*f*, *g*)-manifold or manifold with (*f*, *g*)-structure. When $\varpi = \rho = -1$, we have almost Hermitian structure in the M_n.

If $\varpi = -1$ ($\varpi = 1$), (*f*, *g*)-structure is called the elliptic (hyperbolic); if $\rho = -1$ ($\rho = +1$), (*f*, *g*)-structure is called of the first (second) kind.

(*f*, *g*) – structure in M_n may be defined by the affinor $f_i^{\ j}$ (1) and exterior 2-form $A = f_{ij} dx^i \wedge dx^j$, when $\rho = -1$, or quadratic form $B = f_{ij} dx^i dx^j$, when $\rho = 1$. In fact, if we have tensor $f_{ij} = \rho f_{ji}$, $\rho = \pm 1$, for which

$$g_{ik} = \varpi f_i^{\ j} f_{jk} \tag{3}$$

is symmetric non-singular tensor, then according to (1) $f_i^{\ j} g_{jk} = f_i^{\ j} \varpi f_j^{\ l} f_{lk} = f_{ik}$ and tensors $f_i^{\ j}$, g_{ij} define (*f*, *g*)-structure. Therefore 2-forms A and B, defined by structural covariant tensor f_{ij}, can be called fundamental form of the (*f*, *g*)-structure.

Suppose that

$$\nabla_i f_j^k = \partial_i f_j^k - \Gamma_{ij}^l f_l^k + \Gamma_{il}^k f_i^l = 0, \tag{4}$$

where ∇ and Γ are correspondingly symbol of covariant differentiation and Christoffel's symbol, defined by Riemannian metric g. So we deal with some analogues of Kahlerian structure.

Let $T(M_n)(x^i, x^{n+i})$ be tangent bundle of M_n, x^{n+i} – coordinates of a vector at the point $M \in M_n$. The matrixes of complete lifts ${}^c f$ and ${}^c g$ of tensors f and g are:

$$\left({}^c f_\alpha^\beta\right) = \begin{pmatrix} f_i^j & x^{n+p}\partial_p f_i^j \\ 0 & f_i^j \end{pmatrix}, \tag{5}$$

$$\left({}^c g_{\alpha\beta}\right) = \begin{pmatrix} x^{n+p}\partial_p g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}, \alpha, \beta, \ldots = 1, \ldots, n, n+1, \ldots 2n. \tag{6}$$

We can easily verify that in the view of (1), (2) tensors ${}^c f$ and ${}^c g$ satisfy the conditions ${}^c f_\alpha^\beta \; {}^c f_\beta^\gamma = \varpi \delta_\alpha^\gamma$, ${}^c f_\alpha^\beta \; {}^c g_{\beta\gamma} = F_{\alpha\gamma} = \rho F_{\gamma\alpha}$, where

$$\left(F_{\alpha\gamma}\right) = \begin{pmatrix} x^{n+p}\partial_p f_{ij} & f_{ij} \\ f_{ij} & 0 \end{pmatrix}. \tag{7}$$

Thus, we have (${}^c f$, ${}^c g$)-structure in the $T(M_n)$ of the same type and kind as in the basic manifold M_n.

We shall prove that from (4) ${}^c\nabla_\alpha \; {}^c f_\beta^\gamma = 0$, where ${}^c\nabla$ is the symbol of covariant differentiation in Riemannian connection, defined by metric ${}^c g$.

At first we find $({}^c g_{\alpha\beta}) = ({}^c g_{\alpha\beta})^{-1}$:

$$\left({}^c g^{\alpha\beta}\right) = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & -x^{n+p}\partial_p g_{kl} g^{ki} g^{lj} \end{pmatrix}, \tag{8}$$

where $g^{ij} g_{jk} = \delta_k^i$. Non-zero Christoffel's symbols $\theta_{\alpha\beta}^\gamma$ of the metric ${}^c g$ in view of (6), (8) and $\nabla_i g_{jk} = 0$ are [2]:

$$\theta_{ij}^k = \theta_{n+i,j}^{n+k} = \theta_{i,n+j}^{n+k} = \Gamma_{ij}^k, \quad \theta_{ij}^{n+k} = x^{n+p}\partial_p \Gamma_{ij}^k. \tag{9}$$

From (4), (5), (9) and $\partial_{n+p} f_i^j = 0$ after some calculations we have for all α, β, γ:

$${}^c\nabla_\gamma \; {}^c f_\alpha^\beta = \partial_\gamma f_\alpha^\beta - \theta_{\gamma\alpha}^\delta f_\delta^\beta + \theta_{\gamma\delta}^\beta f_\alpha^\delta = 0. \tag{10}$$

Using (2), (7) and

$$\nabla_p f_{ij} = \partial_p f_{ij} - \Gamma_{p\,i}^{m} f_{mj} - \Gamma_{p\,j}^{m} f_{im} = 0, \tag{11}$$

we shall find the fundamental form of (${}^c f$, ${}^c g$)-structure in $T(M_n)$.

If $\rho = -1$

$$(F_{\alpha\beta} = -F_{\beta\alpha}, f_{ij} = -f_{ji}),$$

$$F_{\alpha\beta}dx^{\alpha} \wedge dx^{\beta} = x^{n+p}\partial_p f_{ij}dx^i \wedge dx^j + f_{ij}dx^i \wedge dx^{n+j} + f_{ij}dx^{n+i} \wedge dx^j$$
$$= x^{n+p}(\Gamma^m_{pi}f_{mj} + \Gamma^m_{pj}f_{im})dx^i \wedge dx^j + 2f_{ij}dx^i \wedge dx^{n+j} = 2x^{n+p}\Gamma^m_{pi}f_{mj}dx^i \wedge$$
$$\wedge dx^j + 2f_{ij}dx^i \wedge dx^{n+j} = 2f_{ij}dx^i \wedge (dx^{n+j} + x^{n+p}\Gamma^j_{pq}dx^q) = 2f_{ij}dx^i \wedge \delta x^{n+j}.$$

Here δ – symbol of covariant differentiation. When $\rho = 1$ ($F_{\alpha\beta} = F_{\beta\alpha}, f_{ij} = f_{ji}$),

$$F_{\alpha\beta}dx^{\alpha}dx^{\beta} = x^{n+p}\partial_p f_{ij}dx^i dx^j + f_{ij}dx^i dx^{n+j} + f_{ij}dx^{n+i}dx^j$$
$$= x^{n+p}(\Gamma^m_{pi}f_{mj} + \Gamma^m_{pj}f_{im})dx^i dx^j + 2f_{ij}dx^i dx^{n+j} = 2x^{n+p}\Gamma^m_{pj}f_{mi}dx^i dx^j +$$
$$+ 2f_{ij}dx^i dx^{n+j} = 2f_{ij}dx^i(dx^{n+j} + x^{n+p}\Gamma^j_{pq}dx^q) = 2f_{ij}dx^i \delta x^{n+j}.$$

Thus, the theorem is true.

THEOREM 1. *Let M_n be (f, g)-manifold for which (4) holds. Complete lift ${}^cf$ (5) of the affinor f with fundamental 2-form $F_{\alpha\beta}dx^{\alpha} \wedge dx^{\beta} = 2f_{ij}dx^i \wedge \delta x^{n+j}$, when $\rho = -1$ ($F_{\alpha\beta}dx^{\alpha}dx^{\beta} = 2f_{ij}dx^i\delta x^{n+j}$, when $\rho = 1$) define in $T(M_n)$ $({}^cf, {}^cg)$-structure of the first (second) kind and (10) holds; ${}^cg_{\alpha\beta} = \omega {}^cf^{\gamma}_{\alpha}F_{\gamma\beta}$ is complete lift of metric g_{ij}.*

2. Now we shall consider the geometry of 2-form $C = 2f_{ij}dx^i\delta x^{n+j}$.

$$C = 2f_{ij}dx^i dx^{n+j} + 2f_{im}\Gamma^m_{pq}x^{n+p}dx^q dx^i$$
$$= x^{n+p}(f_{im}\Gamma^m_{pj} + f_{jm}\Gamma^m_{pi})dx^i dx^j + f_{ij}dx^i dx^{n+j} + \rho f_{ij}dx^{n+i}dx^j = {}^*F_{\alpha\beta}dx^{\alpha}dx^{\beta},$$

$$({}^*F_{\alpha\beta}) = \begin{pmatrix} x^{n+p}(f_{im}\Gamma^m_{pj} + f_{jm}\Gamma^m_{pi}) & f_{ij} \\ \rho f_{ij} & 0 \end{pmatrix}. \quad (12)$$

Obviously ${}^*F_{\alpha\beta} = {}^*F_{\beta\alpha}$. When $\rho = 1$, ${}^*F_{\alpha\beta} = F_{\alpha\beta}$, but ${}^*F_{\alpha\beta} \neq F_{\alpha\beta}$, if $\rho = -1$. Symmetric tensor ${}^*F_{\alpha\beta}$ (12), defined by 2-form C, may be regarded as the new metric in $T(M_n)$. It follows from (3), (4), (5), (12) ${}^cf^{\beta}_{\alpha}{}^*F_{\beta\gamma} = H_{\alpha\gamma} = \rho H_{\gamma\alpha}$, where

$$(H_{\alpha\gamma}) = \omega\begin{pmatrix} x^{n+p}(\Gamma^m_{pk}g_{im} + \rho g_{mk}\Gamma^m_{pi}) & g_{ik} \\ \rho g_{ik} & 0 \end{pmatrix}. \quad (13)$$

Thus, tensors ${}^cf^{\beta}_{\alpha}$ (5) and ${}^*F_{\alpha\beta}$ (12) define in $T(M_n)$ $({}^cf, {}^*F)$-structure of the same type and kind as in the basic manifold M_n.

When $\rho = 1$, from (6), (13) $H_{\alpha\gamma} = \omega^c g_{\alpha\gamma}$ and after some calculations we have $H_{\alpha\gamma} dx^\alpha dx^\beta = 2\omega g_{ij} dx^i \delta x^j$; similarly $H_{\alpha\gamma} dx^\alpha \wedge dx^\gamma = 2\omega g_{ij} dx^i \wedge \delta x^{n+j}$, when $\rho = -1$.

We shall prove the equality ${}^*\nabla_\alpha {}^c f^\gamma_\beta = 0$. Here ${}^*\nabla$ – symbol of covariant differentiation in Riemannian connection defined by metric ${}^*F_{\alpha\beta}$ (12).

At first we find the matrix

$$({}^*F^{\alpha\beta}) = ({}^*F_{\alpha\beta})^{-1} = \begin{pmatrix} 0 & \omega\rho f^{jk} \\ \omega f^{jk} & -\omega x^{n+p}(f^{js}\Gamma^k_{ps} + f^{ks}\Gamma^j_{ps}) \end{pmatrix}, \tag{14}$$

where

$$f^{jk} = f^k_i g^{ij} = f_{im} g^{mk} g^{ij}, f_{ij} f^{jk} = \omega \delta^k_i. \tag{15}$$

Non-zero Christoffel's symbols $\Phi^\gamma_{\alpha\beta}$ of metric ${}^*F$ due to (11), (12), (13) are:

$$\Phi^k_{ij} = \Phi^{n+k}_{i,n+j} = \Phi^{n+k}_{n+i,j} = \Gamma^k_{ij},\ \Phi^{n+k}_{ij} = \frac{1}{2} x^{n+p}(R^k_{ijp} + f_{is} f^{kl} R^s_{jlp} + f_{js} f^{kl} R^s_{ilp}), \tag{16}$$

where $R^k_{ijp} = \partial_i \Gamma^k_{jp} - \partial_j \Gamma^k_{ip} + \Gamma^k_{il}\Gamma^l_{jp} - \Gamma^k_{jl}\Gamma^l_{ip}$ – curvature tensor of M_n.

From (4), (5), (15), (16) after some calculations we get ${}^*\nabla_\alpha {}^c f^\alpha_\beta = 0$.

Thus, we have

THEOREM 2. *Let M_n be (f, g)-manifold and (3) holds. The affinor ${}^c f$ (5) and 2-form $2 f_{ij} dx^i \delta x^{n+j} = {}^*F_{\alpha\beta} dx^\alpha dx^\beta$ define in $T(M_n)$ $({}^c f, {}^*F)$-structure of the same kind and type as in the basic manifold M_n and ${}^*\nabla_\alpha {}^c f^\alpha_\beta = 0$. Structural covariant tensor $H_{\alpha\gamma}$ (13) of $({}^c f, {}^*F)$-structure and the metric g_{ij} are related by equality of 2-forms $H_{\alpha\beta} dx^\alpha \wedge dx^\beta$ and $2\omega g_{ij} dx^i \wedge \delta x^{n+j}$, when $\rho = -1$, and by equality $H_{\alpha\beta} dx^\alpha dx^\beta = 2\omega g_{ij} dx^i \delta x^{n+j}$, when $\rho = 1$.*

3. Let us consider in $T(M_n)$ the Sasaki metric $ds^2 = g_{ij} dx^i dx^j + g_{ij} \delta x^{n+i} \delta x^{n+j}$, which matrix has the form [3]:

$$(S_{\alpha\beta}) = \begin{pmatrix} g_{ij} + g_{ml}\Gamma^m_{pi}\Gamma^l_{qj} x^{n+p} x^{n+q} & g_{mj}\Gamma^m_{pi} x^{n+p} \\ g_{im}\Gamma^m_{pj} x^{n+p} & g_{ij} \end{pmatrix}. \tag{17}$$

Together with complete lift ${}^{c}f_{\alpha}^{\beta}$ (5) of affinor $f_i^{\ j}$ this metric defines in $T(M_n)$ $({}^{c}f, S)$-structure of the same type and kind as in the basic manifold M_n, because ${}^{c}f_{\alpha}^{\beta}\ {}^{c}f_{\beta}^{\gamma} = \omega\delta_{\alpha}^{\gamma}$, ${}^{c}f_{\alpha}^{\beta} S_{\beta\gamma} = K_{\alpha\gamma} = \rho K_{\gamma\alpha}$, where from (2), (5), (17)

$$(K_{\alpha\gamma}) = \begin{pmatrix} f_{ij} + f_{lm}\Gamma_{pi}^{l}\Gamma_{qj}^{m}x^{n+p}x^{n+q} & f_{mj}\Gamma_{pi}^{m}x^{n+p} \\ f_{im}\Gamma_{pj}^{m}x^{n+p} & f_{ij} \end{pmatrix}. \tag{18}$$

We shall prove again ${}^{s}\nabla_{\alpha}{}^{c}f_{\beta}^{\gamma} = 0$, when $\nabla_i f_j^k = 0$, where ${}^{s}\nabla$ – symbol of the covariant differentiation in Riemannian connection defined by metric ${}^{s}g$. At first we find the matrix [3]:

$$(S_{\alpha\beta}^{\gamma})^{-1} = (S^{\alpha\beta}) = \begin{pmatrix} g^{ij} & -g^{mj}\Gamma_{pm}^{i}x^{n+p} \\ -g^{im}\Gamma_{pm}^{j}x^{n+p} & g^{ij} + g^{ml}\Gamma_{pm}^{i}\Gamma_{gl}^{j}x^{n+p}x^{n+q} \end{pmatrix}. \tag{19}$$

Non-zero Christoffel's symbols $Q_{\alpha\beta}^{\gamma}$ of metric $S_{\alpha\beta}$ (17) by (19), (11) are:

$$\begin{aligned} Q_{jk}^{i} &= \Gamma_{jk}^{i} - \frac{1}{2}(R_{jmp}^{i}\Gamma_{rk}^{m} + R_{kmp}^{i}\Gamma_{ij}^{m})x^{n+p}x^{n+r}, \\ Q_{n+j,k}^{i} &= -\frac{1}{2}R_{kjp}^{i}x^{n+p}, Q_{n+j,k}^{n+i} = \Gamma_{jk}^{i} + \frac{1}{2}\Gamma_{pm}^{i}R_{kjr}^{m}x^{n+r}x^{n+p}, \\ Q_{jk}^{n+i} &= \frac{1}{2}(R_{jkp}^{i} + R_{kjp}^{i} + 2\partial_p\Gamma_{jk}^{i})x^{n+p} + \frac{1}{2}\Gamma_{mp}^{i}(R_{jlr}^{m}\Gamma_{sk}^{l} + R_{klr}^{m}\Gamma_{sj}^{l})x^{n+s}x^{n+r}x^{n+p}. \end{aligned} \tag{20}$$

From (4), (5), (20) after some calcululations ${}^{s}\nabla_{\alpha}{}^{c}f_{\beta}^{\gamma} = 0$.

Fundamental 2-form of $({}^{c}f, S)$-structure, denote by $\tilde{A}$, when $\rho = -1$.

$$\tilde{A} = F_{\alpha\beta}dx^{\alpha} \wedge dx^{\beta} = f_{ij}dx^{i} \wedge dx^{j} + f_{lm}\Gamma_{pi}^{l}\Gamma_{qj}^{m}x^{n+p}x^{n+q}dx^{i} \wedge dx^{j}$$

$$f_{mj}\Gamma_{pi}^{m}x^{n+p}dx^{i} \wedge dx^{n+j} + f_{im}\Gamma_{pj}^{m}x^{n+p}dx^{n+i} \wedge dx^{j} + f_{ij}dx^{n+j} \wedge dx^{n+j} = f_{ij}dx^{i} \wedge dx^{j}$$

$$f_{ij}(dx^{n+i} + \Gamma_{pm}^{i}x^{n+p}dx^{m}) \wedge (dx^{n+j} + \Gamma_{qs}^{j}x^{n+q}dx^{s}) = f_{ij}dx^{i} \wedge dx^{j} + f_{ij}\delta x^{n+i} \wedge \delta x^{n+j}.$$

When $\rho = 1$, similarly $\tilde{B} = K_{\alpha\beta}dx^{\alpha}dx^{\beta} = f_{ij}dx^{i}dx^{j} + f_{ij}\delta x^{n+i}\delta x^{n+j}$.

Thus, the following theorem is true.

THEOREM 3. *Let M_n be (f, g)-manifold, for which (3) holds. Complete lift ${}^{c}f$ (5) of the affinor f together with fundamental 2-form $\tilde{A}$ (when $\rho = -1$) or $\tilde{B}$ (when $\rho = 1$) define in the tangent bundle $T(M_n)$ $({}^{c}f, S)$-structure of the same type and kind as in the basic manifold M_n and equality ${}^{s}\nabla_{\alpha}{}^{c}f_{\beta}^{\gamma} = 0$ holds, where $S_{\alpha\beta} = \omega{}^{c}f_{\alpha}^{\gamma}K_{\gamma\beta}$ is Sasaki metric.*

When $\varpi = \rho = -1$, the (f, g)-manifold M_n is Kahlerian and the geometry of the tangent bundle $T(M_n)$ is Kahlerian geometry, defined by fundamental exterior 2-forms $2f_{ij}dx^i \wedge \delta x^j$, $f_{ij}dx^i \wedge dx^j + f_{ij}\delta x^{n+i} \wedge \delta x^{n+j}$ or quadratic form $2f_{ij}dx^i \delta x^{n+j}$.

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On the structure of tangent bundle of (f, g)-manifold defined by 2-forms

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Tiriamos (f, g)-daugdaros M_n, analogiškos Kelerio daugdaroms. Įrodyta, jog liečiamajame pluošte $T(M_n)$ afinoriaus f pilnas liftas ${}^c f$ kartu su 2-formomis, kurias apibrėžia daugdaros M_n struktūrinis kovariantinis tenzorius, indukuoja tokią pat struktūrą kaip ir bazės struktūra. Rasti ryšiai tarp daugdarų M_n ir $T(M_n)$ struktūrinių tenzorių.