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Tableaus for finitely-valued modal propositional logics

J. Sakalauskaitė (MII)

Introduction

Finitely-valued logics have numerous applications to computer science This supports the interest of investigation of automated theorem proving for finitely-valued logics.

In this paper we investigate tableau-based theorem proving for some of the finitely-valued modal propositional logics. We consider finitely-valued counterparts of well-known two-valued modal logics K, $K4$, T, B, $S4$, $S5$ (see, e.g., [1]). Let $L \in \{K, K4, T, B, S4, S5\}$. For L we have a family of finitely-valued counterparts. A counterpart of L is determined by the triple $(T, f, f_\Box)$, where T is the set of truth values, f denotes the set of functions which determine semantics of propositional connectives and $f_\Box$ is a function which is used to define the semantics of modality $\Box$. The definition of finitely-valued counterparts of modal logic L presented in this paper is rather general. It covers many examples of finitely-valued modal logics which have been considered before ([2], [5]–[10]). For logic K such counterparts with some rectrictions on the function $f_\Box$ have been considered by Takano in [9], where the cut free sequent calculi for them are constructed.

In this paper we introduce the formal proof procedures which are called the prefixed signed tableau systems for the each finitely-valued counterpart of logic L, where $L \in \{K, K4, T, B, S4, S5\}$ and prove soundness and completeness theorems for these systems. The prefixed tableau procedures presented in this paper include two devices, i.e. formulas are equiped with prefixes as well as with sets of truth values called signs. Prefixes are well-known in two-valued modal case (see [1]). Sets as signs are familiar in the area of many-valued logics ([3]).

The paper is organized as follows. In section 2 we introduce the syntax and semantics of the finitely-valued modal logics. In section 3 we present the prefixed signed tableau systems for these logics, prove soundness and completeness of the systems.

Syntax and semantics

We put T be a finite set and use T as the set of *truth values*. We let λ, μ, ν, ... denote truth values. We fix a set of propositional connectives with their arities. *Formulas* are constructed from propositional variables by means of propositional connectives and the necessity operator $\Box$.

We assume that for each propositional connective F with arity $\alpha(F) \geqslant 0$ the truth function $f_F : T^{\alpha(F)} \to T$ is fixed. By f we denote the set of functions $\{f_F : F$ is a propositional connective}. Let $P(T)$ denotes the power set of T. We also assume that a function $f_{\Box} : P(T) \to T$ is fixed.

We consider below finitely-valued modal logics $L(T, f, f_{\Box})$, $L \in \{K, K4, T, B, S4, S5\}$. Instead of $L(T, f, f_{\Box})$ we simply write L.

Definition. Pair SA, where $S \subseteq T, S \neq \emptyset$, A is a formula, is called a *signed formula.*

Definition. A *prefix* is a finite sequence of positive integers. A *prefixed signed formula* $\sigma : SA$ is a prefix σ followed by a signed formula SA.

We will systematically use σ, σ', etc. for prefixes throughout this paper.

The idea is, we will interpret prefixes as naming worlds in some model. $\sigma : SA$ means that under this model A is forced to have value from S in the world σ names.

Definition. Suppose that W is a nonempty set (set of worlds), R is a binary relation on W. We call the triplet (W, R, v) a *Kripke structure,* if v is the mapping which assigns a truth value to each pair of a propositional variable and an element of W.

v is extended to arbitrary formulas by recursion as follows:

$$v(F(A_1, \dots, A_{\alpha(F)}), \omega) = f_F(v(A_1, \omega), \dots, v(A_{\alpha(F)}, \omega));$$
$$v(\Box A, \omega) = f_{\Box}(\{v(A, \omega_1) | \omega R \omega_1\}).$$

Definition. Models of K are nothing but the Kripke structures; whereas a *model of* T, $K4$, B, $S4$ is a Kripke structure (W, R, v) such that R is reflexive, transitive, reflexive and symmetric, reflexive and transitive, respectively. A *model* of $S5$ is a Kripke structure (W, R, v), where R is reflexive, symmetric and transitive.

Let L be one of the logics we are considering.

Definition. A signed formula SA is *L-satisfiable* if there exists a model (W, R, v) of L and a world $\omega \in W$ such that $v(A, \omega) \in S$. A signed formula SA is *L-valid* if for each model (W, R, v) of L and for each world $\omega \in W$ $v(A, \omega) \in S$.

Let σ be an arbitrary prefix.

Definition. We say the relation of *accessibility from* on prefixes satisfies:

1) the *general* condition if σn is accessible from σ for every integer n;

2) the *reverse* condition if σ is accessible from σn for every integer n;

3) the *identity* condition if σ is accessible from σ;

4) the *transitivity* condition if the sequence $\sigma\sigma'$ is accessible from σ for every non-empty sequence σ'.

For the various logics we are considering, the conditions which the accessibility relation on prefixes satisfies are given in the following chart.

Logic conditions

K	general
$K4$	general, transitivity
T	general, identity
B	general, identity, reverse
$S4$	general, identity, transitivity
$S5$	no special conditions, any prefix is accessible from any other

Let L be a logic we are considering. Let $\mathcal{A}$ be a set of prefixed signed formulas and let $\mathcal{M} = (W, R, v)$ be a L-model.

Definition. By an *L-interpretation* of $\mathcal{A}$ in the model $\mathcal{M}$ we mean a mapping I from the set of prefixes that occur in $\mathcal{A}$ to W such that if a prefix τ is L-accessible from a prefix σ, then $I(\sigma)RI(\tau)$. $\mathcal{A}$ is *L-satisfiable under the L-interpretation I* if for each $\sigma : SA \in \mathcal{A}$ $v(A, I(\sigma)) \in S$. $\mathcal{A}$ is *L- satisfiable* if $\mathcal{A}$ is L-satisfiable under some L-interpretation.

Loosely, a set of prefixed signed formulas is L-satisfiable if it partially describes some model of L.

By $rg(f_F), rg(f_\Box)$ we denote the range of the functions f_F, $f_\Box$, respectively.

We put by definition $f_\Box^-(S) = \{S_1 \subseteq T \mid f_\Box(S_1) \in S\}$.

LEMMA. *Let* $A = \sigma : SF(A_1, \dots, A_m)$, $m \geq 1, S \subseteq T, S \neq \emptyset$ *be a prefixed signed formula such that* $S \cap rg(f_F) \neq \emptyset$. *Then there are a number* M, *index sets* $\mathcal{I}_1, \dots, \mathcal{I}_M$ *and signs* S_{rs}, $1 \leq r \leq M$ *such that A is L-satisfiable under L-interpretation I iff there exists* $1 \leq r \leq M$ *such that the set* $C_r = \{\sigma : S_{rs}A_s | s \in \mathcal{I}_r\}$ *is L-satisfiable under L-interpretation I.*

Proof. Using the definition of L-satisfiability.

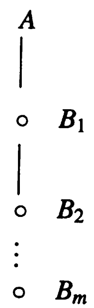
Let $C_1, \dots, C_M$ be sets of prefixed signed formulas as in lemma 8. Following [4] we call the set $\{C_r | 1 \leq r \leq M\}$ a *sets-as-signs DNF representation* of the formula $\sigma : SF(A_1, \dots, A_m)$.

Prefixed tableaus

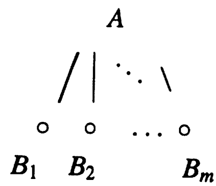
We will consider informally the notions of trees, branches, nodes, etc. We consider the following symbols:

$$\frac{A}{B_1, B_2, \dots, B_m} \quad \text{and} \quad \frac{A}{B_1 + B_2 + \dots + B_m}$$

as denoting trees of the following form, respectively:



and



and we will abbreviate those symbols by the following expressions:

$$\frac{A}{\{B_i \mid i \leqslant m\}} \qquad \frac{A}{+\{B_i \mid i \leqslant m\}}$$

We define tableau rules as follows.

For each prefixed signed formula $A = \sigma : SF(A_1, \ldots, A_m)$, where F is an m-ary propositional connective, $S \subseteq T$, $S \neq \emptyset$, $S \cap rg(f_F) \neq \emptyset$, we define the rule (following [4]) as follows. Let $\{C_r | 1 \leq r \leq M\}$ be a sets as signs DNF representation of the formula A.

$$\frac{\sigma : SF(A_1, \ldots, A_m)}{C_1 + \ldots + C_M}$$

To present rules for modalized formulas we need the following definitions.

Definition. We say a prefix σ is *used* on a tableau branch if $\sigma : Z$ occurs on the branch for some signed formula Z. We say a prefix σ is *unrestricted* on a tableau branch if σ is not an initial segment (proper or otherwise) of any prefix used on the branch.

Definition. A prefixed signed formula which occurs over the line of a rule is called the *premise* of the rule. Let $\mathcal{B}_1 + \ldots + \mathcal{B}_m$, $m \geq 1$ be the expression below the line in a tableau rule. We say that B_j, $1 \leqslant j \leqslant m$ is a $j-$th *consequence* of this rule.

Definition L.et a branch θ contains the premise of a tableau rule. Let this rule has $m, m \geq 1$ consequences. Let the branch θ_j be obtained from θ adding the j-th consequence of this rule, $1 \leq j \leq m$. We call θ_j the *j-th branch* (obtained by this rule for this premise).

We present the tableau rules for modalized formulas for logic L. Each of these rules has conditions which depend on a branch which is supposed to be extended by this rule.

Let θ be a branch such that θ contains the premise of the rule (which is defined below) and θ will be extended by this rule. Let there exists a subset $S', S' \neq \emptyset$ of T such that $S' \in f_{\Box}^{-}(S)$. Let either $\emptyset \notin f_{\Box}^{-}(S)$ and $f_{\Box}^{-}(S) = \{S_1, \ldots, S_m\}$, $m \geq 1$ or let $f_{\Box}^{-}(S) = \{ptyset, S_1, \ldots, S_m\}$, $m \geq 1$ and there is a prefix σ' on θ which is L-accessible from σ. Let $S_j = \{\nu_{j1}, \ldots, \nu_{jk_j}\}$, $1 \leq j \leq m$. Let $\{\sigma n_1, \ldots, \sigma n_{k_j} | 1 \leq j \leq m\}$ be unrestricted prefixes on θ. Then the rule $\Box_1$ is defined as follows.

$$(\Box_1) \ \frac{\sigma : S\Box A}{+\{\sigma n_1 : \nu_{j1}A, \ldots, \sigma n_{k_j} : \nu_{jk_j}A | 1 \leq j \leq m\}}.$$

Definition. Let θ_j, $1 \leq j \leq m$ be the j-th branch obtained from θ by application of the rule $(\Box_1)$ to a formula $\sigma : S\Box A$. Let θ'_j be θ_j or a branch which extends θ_j. We call θ'_j the *S_j-th branch* for the formula $\sigma : S\Box A$. We call prefixes $\sigma n_1, \ldots, \sigma n_{k_j}$ on θ'_j *prefixes of j- th consequence* of the rule $(\Box_1)$ for the premise $\sigma : S\Box A$.

Let the branch θ be the S_j-th branch for a formula $\sigma : S\Box A$ for some j. Let $\{S'_1, \ldots, S'_m\}$ be the set of nonempty subsets of T which belong to $f_{\Box}^{-}(S)$ such that $S_j \subset S'_i$, $1 \leq i \leq m$ (note that this set can be empty). Let $S'_i = S_j \cup \{\nu_{i1}, \ldots, \nu_{ik_i}\}$, $1 \leq i \leq m$. Let $\{\sigma n_1, \ldots, \sigma n_{k_i} |, 1 \leq i \leq m\}$ be unrestricted prefixes on θ. Let σ' be a prefix on θ which is L-accessible from σ and σ' is not a prefix of the j-th consequence of the rule $(\Box_1)$ for the premise $\sigma : S\Box A$. Then the rule $(\Box_{2,j})$ is defined as follows.

$$(\Box_{2,j}) \ \frac{\sigma : S\Box A}{\begin{array}{c}\sigma' : S_jA + \{\sigma' : S'_1A, \sigma n_1 : \nu_{11}A, \ldots, \sigma n_{k_1} : \nu_{1k_1}A\}+, \ldots, \\ +\{\sigma' : S'_mA, \sigma n_1 : \nu_{m1}A, \ldots, \sigma n_{k_m} : \nu_{mk_m}A\}\end{array}}$$

Definition. A L-tableau for a signed formula SA is any tree whose first node is the formula $1 : SA$ and those next nodes are obtained according to tableau rules for logic L.

Definition. A set $\mathcal{A}$ of prefixed signed formulas is *closed* if one of the following conditions holds:

1) in $\mathcal{A}$ there are prefixed signed formulas $\sigma : S_1A, \ldots, \sigma : S_mA$ such that $\cap_{i=1}^{m} S_i = \emptyset$;

2) in $\mathcal{A}$ there is a prefixed signed formula $A = \sigma : SF(A_1, \ldots, A_m)$ such that $S \cap rg(f_F) = \emptyset$ (i.e. no tableau rule with the premise A is defined);

3) in $\mathcal{A}$ there is a prefixed signed formula $A = \sigma : S\Box A$ such that $S \cap rg(f_{\Box}) = \emptyset$ (i.e. no tableau rule with the premise A is defined).

4) in $\mathcal{A}$ there is a prefixed signed formula $\sigma : S\Box A$ such that $f_{\Box}^{-}(S) = \{ptyset\}$ and there exists a prefix σ' which is L-accessible from σ.

Definition. A tableau branch is *closed* if it contains a closed set of formulas. A tableau is *closed* if each branch of it is closed.

Let L be a logic we are considering. Similar as in two-valued modal case (see [1]) the following theorems are proved.

THEOREM (soundness). *If there exists a closed L-tableau for a signed formula SA, then the formula SA is L-unsatisfiable.*

THEOREM (completeness). *If a signed formula SA is L-valid, then there exists a closed L-tableau for formula S_1A where $S_1 = T - S$.*

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Lentelės daugiareikšmėms modalinėms propozicinėms logikoms

J. Sakalauskaitė

Pateiktos lentelių sistemos daugiareikšmėms modalinėms propozicinėms logikoms. Šioms sistemoms įrodytos neprieštaringumo ir pilnumo teoremos.