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Simple proof of the second order renewal theorem

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Let $X_1, X_2, \ldots$ be i.i.d. possitive random variables with common distribution F. We assume, throughout this paper, that F is non-singular. Suppose that $0 < a^{-1} = \mathbf{E}X_1 < \infty$. Let $S_0 = 0$; $S_n = X_1 + \cdots + X_n$, $n > 0$, be so-called random walk. One of the central objects in renewal theory is the family $\mathcal{N}(t) = \inf\{n \geqslant 1\colon\ S_n > t\}$, $t \geqslant 0$, of first passage times for $S_n, n \geqslant 0$. The expectation $U(t) = \mathbf{E}\mathcal{N}(t), t \geqslant 0$, is the so-called renewal function.

A number of authors have investigated the asymptotic behaviour of $U(t)$ as $t \to \infty$ (see [1] and references contained therein). Sufficiently good estimates of hight order $U(t)$ asymptotics are hardly obtainable. In this paper, we present a simple proof of the second order renewal theorem, moreover, here obtained the estimates are valid for a large class of subexponential distributions.

Definition 1. The distribution function F belongs to the subexponential class S, if its tail $\bar{F} := 1 - F$ satisfies $\lim_{t\to\infty} \overline{F * F}(t)/\bar{F}(t) = 2$, where $*$ denotes the Stieltjes convolution of F with itself.

By $\mathcal{F}$ we denote a set of all positive and measurable functions, defined on $[0, \infty)$. For any two functions h and g of $\mathcal{F}$ we define their Lebesque convolution $h \oplus g$ by

$$h \oplus g(t) = \int_0^t h(t-u)g(u)du, \quad t \geqslant 0.$$

In [2] the class OA of functions $g \in \mathcal{F}$ was defined.

We say that $g \in OA$ if

$$g \oplus g(t) = O(1)g(t)\int_0^t g(u)du.$$

Definition 2. he distribution function $F \in S$ belongs to the class OSD, if $\bar{F} \in OA$. Let us define the integrated tail distribution F_1 of F, i.e.,

$$F_1(t) = a\int_0^t \bar{F}(u)du, \quad t \geqslant 0.$$

Then the next result is true.

THEOREM. *If $F_1 \in OSD$, then as $t \to \infty$*

$$U(t) - at - a\int_0^t \bar{F}_1(u)du = O(1)\bar{F}_1(t)\int_0^t \bar{F}_1(u)du. \tag{1}$$

Proof. Let us define $Z(t) = U(t) - at$, $t \geqslant 0$. From Theorem 3.1.11 in [1] it follows that if $F_1 \in OSD$, then

$$A := \sup_{t \geqslant 0} \left| Z(t+1) - Z(t) \right| / \bar{F}_1(t+1) < \infty. \tag{2}$$

We have

$$U(t) - at - a\int_0^t \bar{F}_1(u)du = \bar{F}_1(t)Z(t) + a\int_0^t \Big(Z(t) - Z(t-y)\Big)\bar{F}(y)dy. \tag{3}$$

First of all we choose a positive integer N large enough and rewrite the term $a\int_0^t (Z(t) - Z(t-y))\bar{F}(y)dy$ as $I + II$, where

$$I = a\int_0^{t-N} \Big(Z(t) - Z(t-y)\Big)\bar{F}(y)dy,$$

$$II = a\int_{t-N}^{t} \Big(Z(t) - Z(t-y)\Big)\bar{F}(y)dy.$$

As to II, we have that $II = O(1)Z(t)\bar{F}(t) = o(1)Z(t)\bar{F}_1(t)$. Next consider I. We have that for y: $k \leqslant y \leqslant k+1$

$$\begin{aligned} Z(t) - Z(t-y) &= Z(t) - Z(t-k) + Z(t-k) - Z(t-y) \\ &= \sum_{r=0}^{k-1} \Big(Z(t-r) - Z(t-r-1)\Big) + Z(t-k) - Z(t-y). \end{aligned}$$

From (2) it follows that

$$\Big|Z(t) - Z(t-k)\Big| \leqslant A\sum_{r=0}^{k-1} \bar{F}_1(t-r) \leqslant A\int_{t-k}^{t} \bar{F}_1(u)du.$$

Using part (b) of Theorem 3.1.11 in [1] we conclude that for $k < t - N$ there exists a constant C such that

$$Z(t-k) - Z(t-y) \leqslant C\bar{F}_1(t-k)(y-k) \leqslant C \int_{t-y}^{t-k} \bar{F}_1(u)du.$$

Hence

$$I = O(1) \int_0^t \Big(\int_{t-y}^t \bar{F}_1(u)du \Big) \bar{F}(y)dy = O(1) \int_0^t \int_{t-y}^t \bar{F}_1(u)du d\bar{F}_1(y)$$

$$= O(1)\bar{F}_1(t) \int_0^t \bar{F}_1(u)du + O(1)\bar{F}_1 \oplus \bar{F}_1(t).$$

Since $F_1 \in OSD$, we obtain that $I = O(1)\bar{F}_1(t) \int_0^t \bar{F}_1(u)du$. Now combine this estimate with the estimate for II. It readily follows that

$$U(t) - at - a\Big(1 - \bar{F}_1(t)\Big)^{-1} \int_0^t \bar{F}_1(u)du = O(1)\bar{F}_1(t) \int_0^t \bar{F}_1(u)du.$$

This completes the proof of the result.

Remark. As follows from Theorem 1.2 in [3] we have that estimate (1) is optimal.

COROLLARY. *If* $\mathbf{E}X_1^2 < \infty$, *then under the assumptions of the Theorem it follows that*

$$U(t) - at - a \int_0^t \bar{F}_1(u)du = O(1)\bar{F}_1(t)$$

as $t \to \infty$.

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Paprastas antros eilės atstatymo teoremos įrodymas

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Darbe pateikiamas paprastas antros eilėes atstatymo teoremos įrodymas plačiai subeksponentinių skirstinių klasei.