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On compound Poisson approximations

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Results on compound Poisson approximations (CPA) are numerous – see, for example, [1]–[8], [11]. In this note we consider the dependence of the accuracy of CPA on the additional finite convolutions. This problem was emphasized in [4]. Apart from the CPA we use signed compound Poisson (SCP) approximations. Results of this paper are related to the results from [1–2, 6, 9–11].

Let $\mathcal{F}$ be the set of all distributions, and let $\mathcal{F}_+$ be the set of all distributions with nonnegative characteristic functions. Let E_a denote the distribution concentrated at a point a, $E \equiv E_0$. Products and powers of measures are defined in the convolution sense: $FG = F * G$, $F^n = F^{*n}$, $F^0 = E$. For any signed measure of bounded variation W we denote by $\exp\{W\} = \sum_{k=0}^{\infty} W^k/k!$ its exponential measure, by $|W| = \sup_x |W\{(-\infty, x)\}|$ the analogue of the uniform distance, $\widehat{W}(t) = \int_{-\infty}^{\infty} \exp\{itx\}\,dW$ its Fourier–Stieltjes transform. Let $\|W\| = W^+\{\mathbf{R}\} + W^-\{\mathbf{R}\}$ denote the total variation norm. Here $W = W^+ - W^-$ is the Jordan-Hahn decomposition of W. Note that, for any distribution $F \in \mathcal{F}$, we have $\|F\| = 1$. Moreover, the total variation norm is equivalent to the supremum over all Borel sets.

The symbol C denotes all absolute positive constants. The symbol θ is used for all quantities satisfying $|\theta| \leqslant 1$.

Definition. Let $\lambda > 0$, $F \in \mathcal{F}$. Then $\exp\{\lambda(F - E)\}$ is called a compound Poisson (CP) distribution.

Definition. Let $\lambda \in \mathbf{R}$, $F \in \mathcal{F}$. Then $\exp\{\lambda(F - E)\}$ is called a signed compound Poisson (SCP) measure.

Obviously, CP distributions form a subset of all SCP measures. Let us define a compound distribution by

$$\psi(F, B) = \sum_{j,k=0}^{\infty} q_{jk} F^j B^k, \quad F, B \in \mathcal{F}, \quad \sum_{j,k=0}^{\infty} q_{jk} = 1, \ 0 \leqslant q_{jk} \leqslant 1. \tag{1}$$

Note that $\psi(F, B)$ can be viewed as a distribution of a random sum of two-dimensional vectors. Set

$$\nu_{10} = \sum_{j,k=0}^{\infty} j q_{jk}, \quad \nu_{01} = \sum_{j,k=0}^{\infty} k q_{jk}, \quad \nu_{20} = \sum_{j,k=0}^{\infty} j(j-1) q_{jk},$$

$$\nu_{02} = \sum_{j,k=0}^{\infty} k(k-1) q_{jk}, \quad \nu_{11} = \sum_{j,k=0}^{\infty} jk q_{jk}.$$

LEMMA 1. *Let ν_{20}, ν_{02} and ν_{11} be finite. Then*

$$\begin{aligned}\psi(F,B) = E + \nu_{10}(F-E) + \nu_{01}(B-E) + W_{20}(F-E)^2 \\ + W_{02}(B-E)^2 + W_{11}(F-E)(B-E).\end{aligned} \tag{2}$$

Here W_{20}, W_{02} and W_{11} are finite measures and

$$\|W_{20}\| \leqslant \nu_{20}/2, \quad \|W_{02}\| \leqslant \nu_{02}/2, \quad \|W_{11}\| \leqslant \nu_{11}. \tag{3}$$

Proof. Expansion (3) can be obtained by many various methods. We give a proof based on the Bergström expansion (see, for example, [6]). We have

$$\begin{aligned}F^jB^k = E + j(F-E) + k(B-E) + \sum_{l=2}^{j} F^{j-l}(l-1)(F-E)^2 \\ + \sum_{l=2}^{k}(l-1)F^jB^{k-l}(B-E)^2 + \sum_{l=1}^{j} F^{j-l}k(F-E)(B-E).\end{aligned} \tag{4}$$

Consequently,

$$W_{20} = \sum_{j,k=0}^{\infty}\sum_{l=2}^{j}(l-1)F^{j-l}, \quad W_{02} = \sum_{j,k=0}^{\infty}\sum_{l=2}^{k}(l-1)F^jB^{k-l}, \quad W_{11} = \sum_{j,k=0}^{\infty} k\sum_{l=1}^{j} F^{j-l}.$$

By the properties of the variation norm $\|F^{j-l}\| = \|B^{k-l}\| = 1$.

Quite similarly we get that

$$\varphi(B) = \sum_{j=0}^{\infty} q_jB^j, \quad \sum_{j=0}^{\infty} q_j = 1, \quad 0 \leqslant q_j \leqslant 1, \ j = 0, 1, \dots$$

can be expanded as

$$\varphi(B) = E + \nu_1(B-E) + \nu_2(B-E)^2/2 + W_3(B-E)^3, \tag{5}$$

whenever $\nu_3 \leqslant \infty$. Here

$$\nu_1 = \sum_{j=0}^{\infty} jq_j, \quad \nu_2 = \sum_{j=0}^{\infty} j(j-1)q_j, \quad \|W_3\| \leqslant \nu_3/6 = \sum_{j=0}^{\infty} j(j-1)(j-2)q_j/6.$$

Note that expansions similar to (5) and (3) can be found, for example, in [8].

Now we can formulate our results. Further on we assume that expansions (3) and (5) hold. Set

$$\varepsilon_1 = \Big(1 + \sum_{j=1}^{n} p_j\Big)/n, \quad \varepsilon_2 = \min\Big(\sum_{j=1}^{n} p_j^2, \sum_{j=1}^{n} p_j^2 / \sum_{j=1}^{n} p_j\Big),$$

$$\varepsilon_3 = \min\Big(\sum_{j=1}^n p_j^2, \sum_{j=1}^n p_j^2 / \Big(\sum_{j=1}^n p_j\Big)^{3/2}\Big),$$

$$\varepsilon_4 = e^{2\nu_{10}+2\nu_{01}}\Big(\frac{\nu_{20}+\nu_{10}^2}{n} + (\nu_{02}+\nu_{01}^2)\min\Big(1, \Big(\sum_{j=1}^n p_j\Big)^{-1}\Big) + \frac{\nu_{11}+\nu_{20}\nu_{02}}{n}\min\Big(1, \Big(\sum_{j=1}^n p_j\Big)^{-1/2}\Big)\Big).$$

THEOREM 1. *Let* $0 \leqslant p_j \leqslant C_0 < 1$, $j = 1, 2, \ldots, n$. *Then*

$$\sup_{F\in\mathcal{F}_+}\sup_{B\in\mathcal{F}}\Big|\prod_{j=1}^n \Big((1-p_j)F + p_jB\Big)\psi(F,B) - \exp\Big\{\Big(\sum_{j=1}^n (1-p_j) + \nu_{10}\Big)(F-E) + \Big(\sum_{j=1}^n p_j + \nu_{01}\Big)(B-E)\Big\}\Big| \leqslant C(\varepsilon_1 + \varepsilon_4), \tag{7}$$

$$\sup_{F\in\mathcal{F}_+}\sup_{B\in\mathcal{F}}\Big|\prod_{j=1}^n \Big((1-p_j)F + p_jB\Big)\psi(F,B) - \exp\Big\{\Big(\sum_{j=1}^n (1-p_j) + \nu_{10}\Big)(F-E) + \Big(\sum_{j=1}^n p_j + \nu_{01}\Big)(B-E) - \sum_{j=1}^n p_j^2 (B-E)^2/2\Big\}\Big| \leqslant C(\varepsilon_2 + \varepsilon_4) \tag{8}$$

and

$$\sup_{F\in\mathcal{F}_+}\sup_{B\in\mathcal{F}}\Big|\prod_{j=1}^n \Big((1-p_j)F + p_jB\Big)\psi(F,B) - \exp\Big\{\sum_{j=1}^n (1-p_j)(F-E) + \sum_{j=1}^n p_j(B-E) - \sum_{j=1}^n p_j^2(B-E)^2/2 + \psi(F,B) - E\Big\}\Big| \leqslant C\Big(\varepsilon_2 + \nu_{10}^2 + \nu_{01}^2 \min\Big(1, \Big(\sum_{j=1}^n p_j\Big)^{-1}\Big)\Big). \tag{9}$$

If $F \equiv E$ then the estimates can be improved.

THEOREM 2. *Let* $0 \leqslant p_j \leqslant C_0 < 1$, $j = 1, 2, \ldots, n$. *Then*

$$\sup_{B\in\mathcal{F}}\Big\|\prod_{j=1}^n \Big((1-p_j)E + p_jB\Big)\varphi(B) - \exp\Big\{\Big(\sum_{j=1}^n p_j + \nu_1\Big)(B-E)\Big\}\Big\| \leqslant C\Big(\varepsilon_2 + (\nu_1^2 + \nu_2)\min\Big(1, \Big(\sum_{j=1}^n p_j\Big)^{-1}\Big)\Big), \tag{10}$$

$$\sup_{B\in\mathcal{F}}\Big\|\prod_{j=1}^n \Big((1-p_j)E + p_jB\Big)\varphi(B) - \exp\Big\{\Big(\sum_{j=1}^n p_j + \nu_1\Big)(B-E) - \sum_{j=1}^n p_j^2(B-E)^2/2\Big\}\Big\| \leqslant C\Big(\varepsilon_3 + (\nu_1^2 + \nu_2)\min\Big(1, \Big(\sum_{j=1}^n p_j\Big)^{-1}\Big)\Big) \tag{11}$$

and

$$\sup_{B\in\mathcal{F}}\Big\|\prod_{j=1}^{n}\Big((1-p_j)E+p_jB\Big)\varphi(B)-\exp\Big\{\sum_{j=1}^{n}p_j(B-E)\\-\sum_{j=1}^{n}p_j^2(B-E)^2/2+\varphi(B)-E\Big\|\Big\}\leqslant C\Big(\varepsilon_3+\nu_1^2\min\Big(1,\Big(\sum_{j=1}^{n}p_j\Big)^{-1}\Big)\Big). \tag{12}$$

Remark. Note that, in (8), (9), (11) and (12), we used the SCP measures.

Example. Let $0\leqslant p_j\leqslant C_0<1$. We have

$$\Big\|\prod_{j=1}^{n}((1-p_j)E+p_jE_1)(E/2+E_3/2)-\exp\Big\{\Big(\sum_{j=1}^{n}p_j+3/2\Big)(E_1-E)\Big\}\Big\|\\\leqslant C\Big(\varepsilon_2+\Big(\sum_{j=1}^{n}p_j\Big)^{-1}\Big). \tag{13}$$

Proofs. Let G_1, G_2, D_1, D_2 be finite measures. Then

$$|G_1G_2-D_1D_2|\leqslant|G_1-D_1|\,\|G_2\|+|D_1(G_2-D_2)|, \tag{14}$$

$$\|G_1G_2-D_1D_2\|\leqslant\|G_1-D_1\|\,\|G_2\|+|D_1(G_2-D_2)|. \tag{15}$$

From [5, 6] we have

$$\Big|\prod_{j=1}^{n}\Big((1-p_j)F+p_jB\Big)-\exp\Big\{\sum_{j=1}^{n}(1-p_j)(F-E)+\sum_{j=1}^{n}p_j(B-E)\Big\}\Big|\leqslant C(\varepsilon_1+\varepsilon_2), \tag{16}$$

$$\Big|\prod_{j=1}^{n}\Big((1-p_j)F+p_jB\Big)-\exp\Big\{\sum_{j=1}^{n}(1-p_j)(F-E)+\sum_{j=1}^{n}p_j(B-E)\\-\sum_{j=1}^{n}p_j^2(B-E)^2/2\Big|\Big\}\leqslant C(\varepsilon_1+\varepsilon_3), \tag{17}$$

$$\Big\|\prod_{j=1}^{n}\Big((1-p_j)E+p_jB\Big)-\exp\Big\{\sum_{j=1}^{n}p_j(B-E)\Big\}\Big\|\leqslant C\varepsilon_2, \tag{18}$$

$$\Big\|\prod_{j=1}^{n}\Big((1-p_j)E+p_jB\Big)-\exp\Big\{\sum_{j=1}^{n}p_j(B-E)-\sum_{j=1}^{n}p_j^2(B-E)^2/2\Big\}\Big\|\leqslant C\varepsilon_3. \tag{19}$$

Moreover, for any D_1, by the properties of metrics

$$
\begin{aligned}
&|D_1(\psi(F,B)-\exp\{\nu_{10}(F-E)+\nu_{01}(B-E)\})| \\
&\quad\leqslant |D_1(W_{20}(F-E)^2+W_{02}(B-E)^2+W_{11}(F-E)(B-E)+(\nu_{10}(F-E) \\
&\quad\quad+\nu_{01}(B-E)\sum_{j=2}^{\infty}(\nu_{10}(F-E)+\nu_{01}(B-E))^{j-2}/j!)| \\
&\quad\leqslant e^{2\nu_{10}+2\nu_{01}}C\Big((\nu_{20}+\nu_{10}^2)|D_1(F-E)^2| \\
&\quad\quad+(\nu_{02}+\nu_{01}^2)|D_1(B-E)^2|+(\nu_{11}+\nu_{10}\nu_{01})|D_1(F-E)(B-E)|\Big),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
|D_1(\psi(F,B)-\exp\{\psi(F,B)-E\})| &\leqslant C|D_1(\psi(F,B)-E)^2| \\
&\leqslant C\nu_{10}^2|D_1(F-E)^2|+C\nu_{01}^2|D_1(B-E)^2|,
\end{aligned}
\tag{21}
$$

$$|D_1(\varphi(B)-\exp\{\nu_1(B-E)\}| \leqslant C\exp\{2\nu_1\}|(\nu_1^2+\nu_2)D_1(B-E)|, \tag{22}$$

$$|D_1(\varphi(B)-\exp\{\varphi(B)-E\})| \leqslant C|\nu_1^2(B-E)^2D_1|. \tag{23}$$

Just like in [6] we can show that

$$\exp\Big\{\sum_{j=1}^{n}p_j(B-E)-\sum_{j=1}^{n}p_j^2(B-E)^2/2\Big\}=\exp\Big\{(1-C_0)\sum_{j=1}^{n}p_j(B-E)/2\Big\}W, \tag{24}$$

where $\|W\|\leqslant C$. Moreover, from [5] we have

$$\sup_{F\in\mathcal{F}_+}\sup_{B\in\mathcal{F}}|(F-E)^k(B-E)^j\exp\{a(F-E)+\lambda(B-E)\}|\leqslant C(k,j)a^{-k}\lambda^{-j/2}, \tag{25}$$

$$\|(B-E)^k\exp\{\lambda(B-E)\}\|\leqslant C(k)\lambda^{-k/2}. \tag{26}$$

Now the proof of theorems follows from (14)–(26).

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Apie sudėtines aproksimacijas

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Tarkime, kad turime pakankamai tikslią gardelinių dydžių sumos sudėtinę puasoninę aproksimaciją. Darbe parodyta, kad tokio tipo aproksimacija išliks tiksli ir prie pradinės sumos pridėjus baigtinį skaičių gana bendrų gardelinių atsitiktinių dydžių.