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Approximation of characteristic functions in R^k spaces

F. Mišeikis (VU)

Let the series $\sum_{j=1}^{\infty} \xi_j$ of independent k-dimentional random variables $\xi_1, \xi_2, \ldots$ weakly converges to a random variable (r.v.) η, which has a nondegenerate covariance matrix $V = Cov(\eta)$. We are assuming, that $E\xi_j = 0$ $(j = 1, 2, \ldots)$ and that moments of required order are existing. Let $B^2 = V^{-1}$ and B is positively definite symmetric matrix. $L(\zeta)$ denotes the distribution of a r.v. ζ and $\widehat{L}(\zeta)$ denotes the characteristic function of this r.v. Let us denote $\widehat{G}_j(t) := \widehat{L}(\xi_j)$ $(j = 1, 2, \ldots)$, $\widehat{Q}(t) := \widehat{L}(B\eta)$ and

$$l_s := \sup_{\|t\|=1} \frac{\sum\limits_{j=1}^{\infty} E|\langle t, \xi\rangle|^s}{(\sum\limits_{j=1}^{\infty} E\langle t, \xi_j\rangle^2)^{\frac{s}{2}}} \quad (s = 3, 4).$$

We have this estimation of the characteristic function $\widehat{Q}(t)$ of the r.v. η:

THEOREM 1. *Let $l_3 < \infty$. For any t under condition $\|t\| \leqslant d l_3^{-1/3}$, $d \in (0, \sqrt{2})$*

$$\left|\widehat{Q}(Bt) - \exp\left\{-\frac{1}{2}\|t\|^2\right\}\right| \leqslant \left(d{\cdot}a(d) + \frac{1}{6}\right) l_3 \|t\|^3 exp\{-b(l_3, d)\|t\|^2\},$$

here

$$a(d) := \frac{1}{4}\sum_{r=2}^{\infty}\frac{1}{r}\left(\frac{d^2}{2}\right)^{r-2}, \qquad b(l_3; d) := \frac{1}{2} - d\left(da(d) + \frac{1}{6}\right) l_3^{\frac{2}{3}}.$$

Proof. Because $Cov(B\eta) = I$, where I is the identity matrix, we have

$$E|\langle t, B\xi_j\rangle|^s \leqslant \sum_{j=1}^{\infty} E|\langle t, B\xi_j\rangle|^s \leqslant l_s \|t\|^s \quad (j = 1, 2, \ldots).$$

From this we conclude the inequalities:

$$|\widehat{G}_j(Bt) - 1| \leqslant \frac{1}{2}E\langle t, B\xi_j\rangle^2 \leqslant \frac{1}{2}(E|\langle t, B\xi_j\rangle|^3)^{\frac{2}{3}} \leqslant \frac{1}{2} l_3^{\frac{2}{3}} \|t\|^2 \leqslant \frac{1}{2}d^2$$

$(j = 1, 2, \ldots)$. So there exists the logarithm $\ln \widehat{G}_j(Bt)$ $(j = 1, 2, \ldots)$, and we are getting

$$\left|\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right| = \left|\sum_{j=1}^{\infty} \ln \widehat{G}_j(Bt) + \frac{1}{2}\|t\|^2\right|$$

$$= \left| \sum_{j=1}^{\infty} \left[\ln(1-(1-\widehat{G}_j(Bt))) + \frac{1}{2} E\langle t, B\xi_j\rangle^2 \right] \right|$$

$$= \left| \sum_{j=1}^{\infty} \left[-\sum_{r=2}^{\infty} \frac{1}{r}(1-\widehat{G}_j(Bt))^r + \widehat{G}_j(Bt) - 1 + \frac{1}{2} E\langle t, B\xi_j\rangle^2 \right] \right|$$

$$\leqslant \sum_{j=1}^{\infty} \left[\frac{1}{4}(E\langle t, B\xi_j\rangle^2)^2 \sum_{r=2}^{\infty} \frac{1}{r}\left(\frac{d^2}{2}\right)^{r-2} + \frac{1}{6} E|\langle t, B\xi_j\rangle|^3 \right]$$

$$\leqslant (d) \sum_{j=1}^{\infty} (E|\langle t, B\xi_j\rangle|^3)^{\frac{4}{3}} + \frac{1}{6} l_3 \|t\|^3 \leqslant a(d) l_3^{\frac{4}{3}} \|t\|^4 + \frac{1}{6} l_3 \|t\|^3$$

$$\leqslant \left(d \cdot a(d) + \frac{1}{6}\right) l_3 \|t\|^3 \leqslant d\left(d \cdot a(d) + \frac{1}{6}\right) l_3^{\frac{2}{3}} \|t\|^2.$$

By using the inequality $|e^x - 1| \leqslant |x| e^{|x|}$ we conclude

$$\left| \widehat{Q}(t) - \exp\left\{ -\frac{1}{2}\|t\|^2 \right\} \right| = \exp\left\{ -\frac{1}{2}\|t\|^2 \right\} \left| \exp\left\{ \ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2 \right\} - 1 \right|$$

$$\leqslant \left(d \cdot a(d) + \frac{1}{6}\right) l_3 \|t\|^3 \exp\left\{ -\frac{1}{2}\|t\|^2 + d\left(d \cdot a(d) + \frac{1}{6}\right) l_3^{\frac{2}{3}} \|t\|^2 \right\}$$

$$= \left(d \cdot a(d) + \frac{1}{6}\right) l_3 \|t\|^3 \exp\{-b(l_3; d)\|t\|^2\}.$$

In the case $L(\xi_j) = L(A^{j-1}\xi)$, $(j = 1, 2, \ldots)$ with linear operator A under conditions $m := \inf_{\|x\|=1} \|Ax\| > 0$, $\|A\| < 1$ we have the estimation:

$$l_s \leqslant \frac{1-m}{1-\|A\|}(1-m)^{\frac{s-2}{2}} \tilde{l}_s,$$

here

$$\tilde{l}_s = \sup_{\|t\|=1} \frac{E|\langle t, \xi\rangle|^s}{(E\langle t, \xi\rangle^2)^{s/2}}.$$

This means, that we have the rate of approximation $\sqrt{1-m}$ in the Theorem 1, if $m \nearrow 1$.

THEOREM 2. *Let $l_4 \leqslant 1$. For any t under condition $\|t\| \leqslant 2^{-1} l_4^{-1/4}$*

$$\left| \widehat{Q}(t) - \exp\left\{ -\frac{1}{2}\|t\|^2 \right\} \left(1 + \frac{i^3}{6} \sum_{j=1}^{\infty} E\langle t, B\xi_j\rangle^3 \right) \right|$$

$$\leqslant 0,174 l_4 \|t\|^4 \exp\left\{ -\frac{1}{2}\|t\|^2 \right\} + [0,018 l_4^2 \|t\|^8 + \frac{1}{36} l_3^2 \|t\|^6] \exp\{-0,383\|t\|^2\}.$$

Proof. In this case we use inequality

$$|\widehat{G}_j(t) - 1| \leqslant \frac{1}{2}(E\langle t, B\xi_j\rangle^4)^{\frac{1}{2}} \leqslant \frac{1}{2} l_4^{\frac{1}{2}} \|t\|^2 \leqslant \frac{1}{8}$$

for getting

$$\left|\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right| \leqslant \frac{1}{4}\sum_{r=2}^{\infty} r^{-1}8^{2-r}\sum_{j=1}^{\infty}(E\langle t, B\xi_j\rangle^2)^2 + \frac{1}{6}l_3\|t\|^3$$

$$\leqslant 0,1325 l_4\|t\|^4 + \frac{1}{6}\|t\|^3 \leqslant 0,117 l_4^{\frac{1}{2}}\|t\|^2.$$

Also we have

$$\left|\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2 - \frac{i^3}{6}\sum_{j=1}^{\infty} E\langle t, B\xi_j\rangle^3\right|$$

$$= \left|\sum_{j=1}^{\infty}\left[\ln(1-(1-\widehat{G}_j(Bt))) + \frac{1}{2}E\langle t, B\xi_j\rangle^2 - \frac{i^3}{6}E\langle t, B\xi_j\rangle^3\right]\right|$$

$$= \left|\sum_{j=1}^{\infty}\left[-\sum_{r=2}^{\infty} r^{-1}(1-\widehat{G}_j(Bt))^r + \widehat{G}_j(Bt) - 1 + \frac{1}{2}E\langle t, B\xi_j\rangle^2 - \frac{i^3}{6}E\langle t, B\xi_j\rangle^3\right]\right|$$

$$\leqslant \sum_{j=1}^{\infty}\left[\frac{1}{4}(E\langle t, B\xi_j\rangle^2)^2\sum_{r=2}^{\infty} r^{-1}8^{2-r} + \frac{1}{24}E\langle t, B\xi_j\rangle^4\right] \leqslant 0,174 l_4\|t\|^4.$$

The next inequalitie's complete proof:

$$\left|\widehat{Q}(t) - \exp\left\{-\frac{1}{2}\|t\|^2\right\}\left(1 + \frac{i^3}{6}\sum_{j=1}^{\infty} E\langle t, B\xi_j\rangle^3\right)\right|$$

$$\leqslant \exp\left\{-\frac{1}{2}\|t\|^2\right\}\left|\exp\left\{\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right\} - \left(1 + \frac{i^3}{6}\sum_{j=1}^{\infty} E\langle t, B\xi_j\rangle^3\right)\right|$$

$$\leqslant \exp\left\{-\frac{1}{2}\|t\|^2\right\}\left|\exp\left\{\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right\} - \left(1 + \left(\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right)\right)\right|$$

$$+ \exp\left\{-\frac{1}{2}\|t\|^2\right\}\left|\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2 - \frac{i^3}{6}\sum_{j=1}^{\infty} E\langle t, B\xi_j\rangle^3\right|$$

$$\leqslant \exp\left\{-\frac{1}{2}\|t\|^2\right\}\left(\frac{1}{2}\left|\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right|^2 \exp\left\{\left|\ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2\right|\right\} + 0,174 l_4\|t\|^4\right)$$

$$\leqslant \exp\left\{-\frac{1}{2}\|t\|^2\right\}\left(\frac{1}{2}(0,1325 l_4\|t\|^4 + \frac{1}{6}l_3\|t\|^3)^2 exp\{0,117 l_4^{\frac{1}{4}}\|t\|^2\} + 0,174 l_4\|t\|^4\right)$$

$$\leqslant 0,174 l_4\|t\|^4\exp\left\{-\frac{1}{2}\|t\|^2\right\} + \left[0,018 l_4^2\|t\|^8 + \frac{1}{36}l_3^2\|t\|^6\right]\exp\{-0,383\|t\|^2\}.$$

In the case $L(\xi_j) = L(A^{j-1}\xi)$ $(j = 1, 2, \ldots)$ with linear operator A under conditions: $m > 0$, $\|A\| < 1$ we have the rate of approximation $1 - m$ in the theorem 2, if $m \nearrow 1$.

REFERENCES

[1] R. N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expansions*, Wiley, 1976.