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Asymptotic expansion in the zones of large deviations in terms of Lyapunov's fractions

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Let $\xi_1, \xi_2, \dots, \xi_n$, $n \geqslant 1$, be an independent random variable (r.v.) with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2 > 0$, $j = 1, 2, \dots, n$.

Set

$$S_n = \sum_{j=1}^n \xi_j, \qquad B_n^2 = \sum_{j=1}^n \sigma_j^2, \qquad Z_n = \frac{S_n}{B_n},$$

$$F_{Z_n}(x) = \mathbf{P}\bigl(Z_n < x\bigr), \qquad p_{Z_n}(x) = \frac{d}{dx} F_{Z_n}(x), \tag{1}$$

$$L_{k,n} = \sum_{j=1}^n \mathbf{E}|\xi_j|^k / B_n^k, \quad k = 1, 2, \dots . \tag{2}$$

The quantity $L_{k,n}$ is called the k^{th} Lyapunov fraction and for $2 \leqslant k \leqslant l$,

$$L_{k,n}^{1/(k-2)} \leqslant L_{l,n}^{1/(l-2)} \tag{3}$$

what is proved in (Saulis, Statulevičius (1989), (1991)).

As it is know, Lyapunov's fractions are convenient for constructing asymptotical expansions (Statulevičius (1965)) for distribution function $F_{Z_n}(x)$ and distribution density function $p_{Z_n}(x)$ of the r.v. Z_n.

It turns out that probabilities of large deviations $\mathbf{P}\bigl(Z_n \geqslant x\bigr)$, $x = x(n) \to \infty$, $n \to \infty$, in the Cramer zone and Linnik power zones can also be investigated in terms of Lyapunov fractions. Thus, probabilities of large deviations in such zones mainly depend not on individual but average properties of summands as emphasized in (Wolf (1975), Rudzkis, Saulis, Statulevičius (1979), (1991)).

The paper is devoted for investigating asymptotic expansions of probability $\mathbf{P}(Z_n \geqslant x)$ as in zones of Cramer as well in Linnik power zones in terms of Lyapunov's fractions.

The cumulant of r.v. Z_n be denoted by $\Gamma_k(Z_n)$ and defined by the formula

$$\Gamma_k\bigl(Z_n\bigr) := \frac{1}{i^k} \frac{d^k}{dt^k} \ln f_{Z_n}(t)\Big|_{t=0}, \tag{4}$$

where

$$f_{Z_n}(t) := \mathbf{E}e^{itZ_n} = \prod_{j=1}^n f_{\xi_j}(t/B_n) \tag{5}$$

a be characteristic function (c.f) of the r.v. Z_n.

We say that Lyapunov's fractions $L_{k,n}$ satisfy (L^*) condition if exist $\gamma \geqslant 0$ and $\tau_n > 0$ such that

$$L_{k,n} \leqslant \frac{(k!)^{1+\gamma}}{\tau_n^{k-2}}, \qquad k = 3, 4, \ldots. \tag{L^*}$$

PROPOSITION 1. *If the r.v. γ_j with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2 > 0$, $j = 1, 2, \ldots, n$, satisfy condition (L^*) with the exponent $\gamma = 0$, then*

$$\left|\Gamma_k(Z_n)\right| \leqslant \frac{k!}{\left(\tau_n/(27|\ln \tau_n|)\right)^{k-2}}, \qquad k = 3, 4, \ldots. \tag{6}$$

If condition (L^*) with the exponent $\gamma > 0$ is fulfilled, then

$$\left|\Gamma_k(Z_n)\right| \leqslant \frac{(k!)^{1+\gamma}}{\left(\tau_n/C_1(\gamma)\right)^{k-2}} \vee \frac{k!}{\tau_n^2\left(\tau_n^{1/(1+2\gamma)}/C_2(\gamma)\right)^{k-2}}, \qquad k = 3, 4, \ldots, \tag{7}$$

where $a \vee b = \max\{a, b\}$.

Full and complicated proof of this statement is in (Rudzkis, Saulis, Statulevičius (1979)), where explicit expressions quantities $C_1(\gamma)$ and $C_2(\gamma)$ are presented.

Suppose that for a r.v. ξ_j with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2 < \infty$, $j = \overline{1, n}$, there exist densities $p_{\xi_j}(x)$ such that

$$\sup_x \; p_{\xi_j}(x) \leqslant C_j \leqslant \infty. \tag{D}$$

In the case ξ_j has density, then $C_j = \infty$ by definition.

Let $\xi_j(h)$, $j = 1, 2, \ldots, n$, be the r.v. with the distribution density

$$p_{\xi_j}(x) := e^{hx} p_{\xi_j}(x) \Big/ \int_{-\infty}^{\infty} e^{hx} p_{\xi_j}(x)\, dx \tag{8}$$

and the characteristic function

$$f_{\xi_j(h)}(t) = \mathbf{E} \exp\left\{it\xi_j(h)\right\}.$$

Put

$$S_n(h) = \sum_{j=1}^{n} \xi_j(h), \qquad Z_n(h) = \frac{S_n(h) - M_n(h)}{B_n(h)}, \tag{9}$$

where

$$M_n(h) = \mathbf{E}S_n(h) = \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\, \Gamma_k(S_n)\, h^{k-1}, \tag{10}$$

$$B_n^2(h) = \mathbf{D}S_n(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \Gamma_k(S_n)\, h^{k-2}, \tag{11}$$

and

$$f_{Z_n(h)}(t) := \mathbf{E}\, e^{itZ_n(h)} = \prod_{j=1}^{n} f_{\xi_j(h)}\big(t/B_n(h)\big). \tag{12}$$

Here $f_{Z_n(h)}(t)$ is the characteristic function of r.v. $Z_n(h)$ conjugate with r.v. Z_n.

The author of this paper has proved that

$$\Gamma_k\big(S_n(h)\big) = \sum_{l=k}^{\infty} \frac{1}{(l-k)!} \Gamma_l\big(S_n\big)\, h^{l-k}, \qquad k = 2, 3, \ldots. \tag{13}$$

Let

$$\tau_n^* = \begin{cases} \dfrac{c\tau_n}{|\ln \tau_n|}, & \gamma = 0, \\ c_\gamma^* \tau_n^{1/(1+2\gamma)}, & \gamma > 0, \end{cases} \tag{14}$$

where explicit expressions quantities c and c_γ^* are presented in (Rudzkis, Saulis, Statulevičius (1979)), and

$$f_{n,\gamma}^*(t) = \begin{cases} \displaystyle\sum_{k=0}^{s} \left(\frac{3}{2}\right)^k \frac{x^k}{k!} f_{Z_n}^{(k)}(t), & \gamma > 0, \\ f_{Z_n(h)}(t), & \gamma = 0. \end{cases} \tag{15}$$

Here $\sqrt{s} \geqslant c_\gamma^* \tau_n^{1/(1+2\gamma)}$ and $f_{Z_n}(t)$, $f_{Z_n(h)}(t)$ are defined by relation (5) and (13).

The estimates (6) and (7) of cumulants of r.v. Z_n and the authors lemma (Saulis (1996)) of asymptotic of distribution function in zones of large deviations for a r.v. with cumulants of regular behaviour enable to obtain the following statement.

Let Θ (with or without an index) denote some variable, not always the same, not exceeding 1 in absolute value.

PROPOSITION 2. *If the r.v. ξ_j with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2 > 0$, $j = 1, 2, \ldots, n$, satisfy condition (L^*), then $\forall l$, $l \geqslant 3$, in the interval*

$$0 \leqslant x < \tau_n^* \tag{16}$$

the relation of large deviation

$$\frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} = \exp\big\{L_{n,m}^*(x)\big\}\bigg\{\frac{\psi(x)}{\psi(u_n)}\bigg(1 + \sum_{\nu=1}^{l-3} L_{n,\nu}(u_n) \tag{17}$$
$$+ \theta_1(x+1)\bigg[\frac{c(l,\gamma,x)}{\big(\tau_n/|\ln \tau_n|\big)^{l-2}} + \frac{285\tau_n^* \exp\big\{-(1 - x/\tau_n^*)\sqrt{\tau_n^*}\big\}}{(1 - x/\tau_n^*)} + \frac{6q}{T_n} + R_{n,\gamma}\bigg]\bigg\}$$

holds.

Here the quantities $c(l,\gamma,x)$ *and* q *are defined by relations (9) and (58) (Saulis (1996));*

$$\psi(x)=\frac{\varphi(x)}{1-\Phi(x)}, \tag{18}$$

where $\varphi(x)$ *and* $\Phi(x)$ *are standard normal density and distribution function;*

$$L^*_{n,m}(x)=\sum_{3\leqslant k<m}\lambda_{k,n}x^k, \qquad m=\begin{cases}(1/\gamma)+l-1, & \gamma>0,\\ \infty, & \gamma=0,\end{cases} \tag{19}$$

where the coefficients $\lambda_{k,n}$ *are expressed in terms of cumulants of the r.v.* Z_n *and are found from the recurrent equations (2.9) (Saulis, Statulevičius (1989), (1991)):*

$$\lambda_{3,n}=\frac{1}{3}\Gamma_3(Z_n),$$
$$\lambda_{4,n}=\frac{1}{24}\big(\Gamma_4(Z_n)-3\Gamma_3^2(Z_n)\big),\ \ldots .$$

Formulas for $L_{n,\gamma}(u_n)$ *are presented in (Saulis (1996)):*

$$L_{n,1}(u_n)=-\frac{1}{3}\Gamma_3(Z_n)\frac{1}{x}+\frac{3}{2}\big(2\Gamma_4(Z_n)-3\Gamma_3^2(Z_n)\big)$$
$$+\frac{1}{48}\big(72\Gamma_5(Z_n)-39\Gamma_3(Z_n)\Gamma_4(Z_n)+267\Gamma_3^3(Z_n)\big)x+\cdots;$$

$$R_{n,\gamma}=\int_{T_{n,\gamma}}^{T_n}\left|f^*_{n,\gamma}(t)\right|\frac{dt}{t}, \tag{20}$$

where $f^*_{n,\gamma}(t)$ *is defined by (15) and*

$$T_{n,\gamma}=(3/8)\big(1-x/\tau_n^*\big)\tau_n^*, \qquad T_n\geqslant T_{n,\gamma}. \tag{21}$$

So, in order to obtain asymptotic expansions in Cramer zone and Linnik power zones of large deviation probabilities $\mathbf{P}(Z_n\geqslant x)=1-F_{Z_n}(x)$ we need to estimate the integral $R_{n,\gamma}$ defined in (20) for $\gamma=0$ and $\gamma>0$.

PROPOSITION 3. *If for a r.v.* ξ_j *with* $\mathbf{E}\xi_j=0$ *and* $\sigma_j^2=\mathbf{E}\xi_j^2>0$, $j=1,2,\ldots,n$, *conditions* (L^*) *with the exponent* $\gamma=0$ *and* (D) *are fulfilled, then*

$$R_{n,\gamma}\leqslant B_1K_n^2\max_{1\leqslant r_i\leqslant n}\prod_{i=1}^{4}C_{r_i}^{1/4}\exp\left\{-\frac{1}{K_n^2}\sum_{j=1}^{n}\frac{1}{C_j^2}\right\}, \tag{22}$$

where

$$K_n=B_n|\ln\tau_n|/\tau_n. \tag{23}$$

PROPOSITION 4. *If r.v. ξ_j satisfies conditions (L^*) with $\gamma > 0$, (D) and*

$$\varliminf_{n\to\infty} \frac{1}{(1 \vee L_{1,n})\tau_n^*} \sum_{k=1}^{n} \frac{1}{(\sigma_k^2 + N_n^2)C_k^2} \geqslant c_1 > 0,$$

$$N_n = 4B_n L_{3,n}, \qquad \tau_n^* = c_\gamma^* \tau_n^{1/(1+2\gamma)},$$

then

$$R_{n,\gamma} \leqslant B_2\big(N_n/T_{n,\gamma}\big) \max_{1\leqslant r_i\leqslant n} \prod_{i=1}^{4} C_{r_i}^{1/4}\Big(1 + \frac{\pi\sigma_{r_i}}{2\sqrt{2}N_n}\Big) \times \exp\Big\{-\frac{1}{4}\sum_{k=1}^{n}\frac{1}{(\sigma_k^2 + N_n^2)C_k^2}\Big\}, \tag{24}$$

where $T_{n,\gamma} = (3/8)\big(1 - x/\tau_n^*\big)\tau_n^*$.

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Asimptotiniai skleidiniai didžiųjų nuokrypių zonose Liapunovo trupmenų terminais

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Tarkime, kad atsitiktiniai dyžiai (at.d.) ξ_j su vidurkiais $\mathbf{E}\xi_j = 0$ ir dispersijomis $\sigma_j^2 = \mathbf{E}\xi_j^2 > 0$, $j = 1, 2, \dots, n$, tenkina sąlygą: $\exists$ dydžiai $\gamma > 0$ ir $\tau_n > 0$ tokie, kad Liapunovo trupmenos

$$L_{k,n} := \frac{\sum_{j=1}^{n} \mathbf{E}|\xi_j|^k}{B_n^k} \leqslant \frac{(k!)^{1+\gamma}}{\tau_n^{k-2}}, \qquad k = 3, 4, \dots, \qquad B_n^2 = \sum_{j=1}^{n} \sigma_j^2. \tag{L^*}$$

Esant patenkintai sąlygai (L^*) ir reikalaujant at. d. ξ_j tankio funkcijos aprėžtumo, darbe gauti $\mathbf{P}(Z_n \geqslant x)$, $Z_n = S_n/B_n$, $S_n = \xi_1 + \xi_2 + \dots + \xi_n$, asimptotiniai skleidiniai didžiųjų nuokrypių zonose $0 \leqslant x < \tau_n^*$, kur

$$\tau_n^* = \begin{cases} c\tau_n/|\ln \tau_n|, & \gamma = 0, \\ c_\gamma^* \tau_n^{1/(1+2\gamma)}, & \gamma > 0. \end{cases} \tag{14}$$