LMD konf. darbai, 24–28
1997, Vilnius

On one additive problem

A. Kačėnas (VU)

1. INTRODUCTION

Let $s = \sigma + it$ be a complex value, $f(s)$ means the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

in the domain of absolute convergence. One of the most important question in the theory of Dirichlet series is to find their order. This problem is open even for the simplest series, namely the Riemann zeta function. One of the possible way to solve this problem, is to consider the mean values

$$\int_0^T | f(\sigma + it) |^k dt$$

or the mean values via Laplace transform

$$\int_0^\infty | f(\sigma + it) |^k e^{-\delta t} dt,$$

where $0 < \sigma < 1$.

To obtain upper bounds or asymptotic formulas for the integrals mentioned above, one need to have the estimates for the multiple sums of coefficients of Dirichlet series. As for example the estimate for the fourth mean value of the Riemann zeta function depends on the asymptotic behaviour of the sum $\sum_{n\leqslant x} d(n)d(n+k)$, where $d(n)$ denotes the divisor function.

The aim of this paper is to obtain the asymptotic formula for the sum

$$\sum_{n\leqslant x} \sigma_{-\delta}^2(n),$$

where $0 < \delta < \frac{1}{2}$, and $\sigma_{-\delta}(n) = \sum_{d|n} d^{-\delta}$. One needs such formula on considering the shifted fourth mean value of the Riemann zeta function

$$\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 |\zeta(\sigma + it)|^2 e^{-\delta t} dt.$$

We will prove the following theorem.

THEOREM. *Let* $0 < \delta < 3/4$. *Then*

$$\sum_{n\leqslant x}\sigma_{-\delta}^2(n) = \zeta^2(1+\delta)\,\zeta(1+2\delta)\frac{x}{\zeta(2+2\delta)}$$
$$+\zeta(1-\delta)\,\zeta(1+\delta)\,\zeta^{-1}(2)\,x^{1-\delta}\left(\frac{2\gamma+\ln x}{1-\delta}-\frac{1}{(1-\delta)^2}\right)$$
$$+\frac{\zeta^{-1}(2)}{1-\delta}x^{1-\delta}\bigg(\zeta'(1-\delta)\zeta(1+\delta)+\zeta(1-\delta)\zeta'(1+\delta)$$
$$-2\zeta(1-\delta)\zeta(1+\delta)\frac{\zeta'(2)}{\zeta(2)}\bigg)$$
$$+\zeta(1-2\delta)\zeta^2(1-\delta)\frac{x^{1-2\delta}}{(1-2\delta)\,\zeta(2-2\delta)}+O(x^{1/2}\log x).$$

Notation: $A, C, c_1, c_2, \ldots$ denote absolute positive constants (not necessarily the same at each occurence; γ means Euler's constant, defined by $\lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n) = 0.5772157\ldots$; $f(x) = O(g(x))$ means $|f(x)| \leqslant Cg(x)$ for $x \geqslant x_0$, some absolute constant C and a positive function $g(x)$; the Vinogradov symbol $f(x) \ll g(x)$ means the same as $f(x) = O(g(x))$; ε is an arbitrarily small positive number, not necessarily the same at each occurence.

2. PRELIMINARY LEMMAS

In this section we give the classical results on the Riemann zeta function, which will be used in the proof of Theorem. There is known that $\zeta(s)$ is analytic function in the whole complex plane except the point $s = 1$, where it has the simple pole with residue equal to 1. Also, there is known that the Riemann zeta function satisfies the following functional equality

$$\zeta(s) = \chi(s)\,\zeta(1-s), \tag{1}$$

where

$$\chi(s) = 2^s\,\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s).$$

LEMMA 1. *Let* $\alpha \leqslant \sigma \leqslant \beta$, *and* $t \to \infty$. *Then*

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{\pi}{4})}\left(1+O\left(\frac{1}{|t|}\right)\right).$$

The proof of this result one can find in [4].

The idea of the proof of Theorem is based on the equality

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^s}, \tag{2}$$

which holds in the half-plane $\sigma > \max\{1, \operatorname{Re} a+1, \operatorname{Re} b+1, \operatorname{Re}(a+b)+1\}$. This equality is due to Ramanujan [3]. Regarding to the above formula, we need the estimate for $1/\zeta(s)$. We will use the following result.

LEMMA 2. *Let* $\sigma \geqslant 1 - \frac{A}{\ln t}$. *Then*

$$\frac{1}{\zeta(s)} = O(\ln t).$$

The proof may be found in [4].

LEMMA 3. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad (\sigma > 1),$$

where $a_n = O\big(\psi(n)\big)$ *for the non-decreasing function* $\psi(x)$, *and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O\left(\frac{1}{(\sigma - 1)^\alpha}\right),$$

as $\sigma \to 1$. *Let also* $c > 0$, $\sigma + c > 1$, x *be non-integer* N *is the nearest integer to* x. *Then*

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+\omega) \frac{x^\omega}{\omega} d\omega + O\left(\frac{x^c}{T(\sigma + c - 1)^\alpha}\right) + \\ + O\left(\frac{\psi(2x) x^{1-\sigma} \ln x}{T}\right) + O\left(\frac{\psi(N)\, x^{1-\sigma}}{T\,|x - N|}\right).$$

This lemma may be treated as Perron's formula. The proof may be found in [4].

LEMMA 4. *Uniformly in* σ,

$$\zeta(\sigma + it) \ll \begin{cases} 1, & \sigma \geqslant 2, \\ \log t, & 1 \leqslant \sigma \leqslant 2, \\ t^{(1-\sigma)/2} \log t, & 0 \leqslant \sigma \leqslant 1, \\ t^{1/2-\sigma} \log t, & \sigma \leqslant 0. \end{cases}$$

The proof may be found in [1].

3. THE PROOF OF THEOREM

Without lost of generality we may admit x being non-integer and the distance from x to the nearest integer being more than 1/4, say. By Lemma 3, we already have that

$$\sum_{n<x} \sigma_{-\delta}^2(n) = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} \zeta(s)\, \zeta^2(s+\delta)\, \zeta(s+2\delta) \frac{x^s\, ds}{s \cdot \zeta(2s + 2\delta)} + O\big(x^\varepsilon\big). \qquad (3)$$

Further, we consider the integral

$$I = \frac{1}{2\pi i}\int_{\mathcal{P}} \zeta(s)\zeta^2(s+\delta)\zeta(s+2\delta)\frac{x^s}{s\cdot\zeta(2s+2\delta)}\,ds, \tag{4}$$

where the countour of integration $\mathcal{P}$ is the rectangle $c \pm ix$, $\frac{1}{2} - \delta \pm ix$, and $c > 1$. By the residue theorem, we obtain that

$$\begin{aligned} I &= \zeta^2(1+\delta)\,\zeta(1+2\delta)\frac{x}{\zeta(2+2\delta)} \\ &+ \zeta(1-\delta)\,\zeta(1+\delta)\,\zeta^{-1}(2)\,x^{1-\delta}\left(\frac{2\gamma+\ln x}{1-\delta} - \frac{1}{(1-\delta)^2}\right) \\ &+ \frac{\zeta^{-1}(2)}{1-\delta}x^{1-\delta}\Big(\zeta'(1-\delta)\zeta(1+\delta) + \zeta(1-\delta)\zeta'(1+\delta) \\ &\qquad\qquad - 2\zeta(1-\delta)\zeta(1+\delta)\frac{\zeta'(2)}{\zeta(2)}\Big) \\ &+ \zeta(1-2\delta)\zeta^2(1-\delta)\frac{x^{1-2\delta}}{(1-2\delta)\,\zeta(2-2\delta)}. \end{aligned} \tag{5}$$

Hence, in view of the estimates (3)–(5) it remains to evaluate the sum

$$\left(\int_{c+ix}^{\frac{1}{2}-\delta+ix} + \int_{1-\delta+ix}^{\frac{1}{2}-\delta-ix} + \int_{\frac{1}{2}-\delta-ix}^{c-ix}\right)\zeta(s)\,\zeta^2(s+\delta)\,\zeta(s+2\delta)\,\frac{x^s\,ds}{s\cdot\zeta(2s+2\delta)} = I_1 + I_2 + I_3.$$

By Lemmas 3 and 4, we have

$$I_1 + I_3 = O\left(x^{\frac{1}{2}}\right). \tag{6}$$

We estimate below the integral I_2. It can be rewritten into the form

$$I_2 = \int_{-x}^{x} \zeta\left(\frac{1}{2}-\delta+it\right)\zeta^2\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+\delta+it\right)\frac{x^{\frac{1}{2}-\delta+it}dt}{\left(\frac{1}{2}-\delta+it\right)\zeta(1+it)}.$$

Applying the functional equality (1), lemmas 1 and 2, we obtain

$$I_2 = O\left(x^{\frac{1}{2}-\delta}\int_{-x}^{x}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left|\zeta\left(\frac{1}{2}+\delta+it\right)\right|^2 t^{\delta-1}\ln t\,dt\right).$$

Then, by Cauchy inequality, we have

$$I_2 = O\left(x^{\frac{1}{2}}\ln x\cdot\sqrt{\int_{-x}^{x}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4\frac{dt}{t}}\cdot\sqrt{\int_{-x}^{x}\left|\zeta\left(\frac{1}{2}+\delta+it\right)\right|^4\frac{dt}{t}}\right).$$

The classical results for the mean values of the Riemann zeta–function, and the result of the author [2] yields

$$\int_{a}^{2a} |\zeta(\sigma+it)|^4 \frac{dt}{t} = O(\log^4 a),$$

uniformly in $\frac{1}{2} \leqslant \sigma < 1$. Therefore

$$I_2 = O\left(x^{\frac{1}{2}} \ln^6 x\right). \tag{7}$$

Whence, by the estimates (2)–(7), the theorem follows.

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Pastaba apie funkcijos $\zeta(s)$ antros eilės vidurkį kritinėje juostoje

A. Kačėnas

Tarkime, $\zeta(s)$ – Rymano dzeta funkcija, $s = \sigma + it$ – kompleksinis kintamasis ir $1/2 < \sigma < 1$. Pažymėkime

$$E_\sigma(T) = \int_0^T |\zeta(\sigma+it)|^2 dt - \zeta(2\sigma)T - (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}.$$

Straipsnyje nagrinėjama Rymano dzeta funkcijos antros eilės vidurkio asimptotika kritinėje juostoje. Naudojantis naujausiais analizinės skaičių teorijos metodais pagerinami funkcijos $E_\sigma(T)$ įverčiai kritinės juostos srityje. Gauti įverčiai yra tolydūs σ atžvilgiu.