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A remark on the universality of the Riemann zeta-function

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Some years ago we observed [5] that the estimate of the rate of convergence in limit theorem for the Riemann zeta-functions in the space of analytic functions allows to effectivize the universality theorem for this function. During the conference on Analytic number theory in Kyoto (May 20–24, 1996). Prof. E. Bombieri called our attention to the importance of the convergence rate problem in functional limit theorems for Dirichlet series. This influenced us to return to the effectivization problem of the universality theorem.

Let $s = \sigma + it$ be a complex variable, and let, as usual, $\zeta(s)$ denote the Riemann zeta-function. In 1975 S. M. Voronin [7] discovered the universality property for $\zeta(s)$. Now this property bears a name of the universality theorem. Roughly speaking the universality theorem asserts that any analytic function with no zeros can be approximated uniformly on compact subsets of the strip $D = \{s \in \mathbb{C},\ \frac{1}{2} < \sigma < 1\}$ by translations of $\zeta(s)$. Here by $\mathbb{C}$ we denote the complex plane.

THEOREM 1. (Voronin). *Let $0 < r < \frac{1}{4}$. Let $f(s)$ be any nonvanishing continuous function on the disk $|s| \leqslant r$ which is analytic in the interior of this disk. Then for every $\varepsilon > 0$ there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon .$$

Proof is given in [7], [4].

As it was observed by B. Bagchi [1] for the proof of the universality theorem for $\zeta(s)$ a limit theorem in the space of analytic functions can be applied. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T], \ldots \right\},$$

where in place of dots we write a condition satisfied by τ, and meas$\{A\}$ denotes the Lebesgue measure of the set A. Moreover, let $\mathcal{B}(S)$ stands for the class of Borel sets

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of the space S. Let P_n and P be probability mesures on $(S, \mathcal{B}(S))$. We recall that P_n converges weakly to P as $n \to \infty$ if

$$\int_S f\, dP_n \longrightarrow \int_S f\, dP, \quad n \to \infty,$$

for every real bounded continuous function f on S. Define on $(H(D), \mathcal{B}(H(D)))$ a probability measure

$$P_T(A) = \nu_T\big(\zeta(s + i\tau) \in A\big).$$

THEOREM 2 (Bagchi). *There exists a probability measure P on $(H(D), \mathcal{B}(H(D)))$ such that the measure P_T converges weakly to P as $T \to \infty$.*

Proof can be found in [1], [6].

In order to apply Theorem 2 for the proof of the universality of $\zeta(s)$ we need the explicit form of the limit measure P. For this aim the following topological structure is used. Let γ denote the unite circle on complex plane, that is $\gamma = \{s \in \mathbb{C},\ |s| = 1\}$, and we set

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime number p. With the product topology and pointwise multiplication Ω is a compact Abelian topological group. Thus there exists the probability Haar measure m on $(\Omega, \mathcal{B}(\Omega))$, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and define on $(\Omega, \mathcal{B}(\Omega), m)$ an $H(D)$–valued random element $\zeta(s, \omega)$ by the formula

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}$$

Note that the latter Euler's product converges uniformly on compact subsets of D for almost all $\omega \in \Omega$. Let P_ζ stands for the distribution of the random element $\zeta(s, \omega)$.

THEOREM 3 (Bagchi). *The limit measure P in Theorem 2 coinsides with the measure P_ζ.*

Proof is given in [1], [6].

Theorem 3 plays the principal role in the proof of the following universality theorem for the Riemann zeta-function.

THEOREM 4. *Let K be a compact subset of the strip D with connected complement. Let $f(s)$ be a nonvanishing continuous function on K which is analytic in the interior of K. Then for every $\varepsilon > 0$*

$$\liminf_{T \to \infty} \nu_T\left(\sup_{s \in K} \big|\zeta(s + i\tau) - f(s)\big| < \varepsilon\right) > 0.$$

Proof of Theorem 4 is given in [5]. It is similar to that of [1], [2] where K was a compact simply connected and locally path connected subset of D.

Now it arises the important problem to effectivize the universality theorem for $\zeta(s)$, that is to indicate an interval $I = I(f, \varepsilon, K)$ containing a real number τ such that

$$\sup_{s\in K} \left|\zeta(s+i\tau) - f(s)\right| < \varepsilon .$$

It turns out that the effectivization of Theorem 4 can be obtained using estimates of the rate of convergence in Theorem 3. The latter approach was proposed in [5].

Let K, $f(s)$ and ε be the same as in Theorem 4. Denote by $A(f, \varepsilon, K)$ a ball with a center f and radius ε in $H(D)$, i.e.

$$A(f, \varepsilon, K) = \left\{g \in H(D)\colon \sup_{s\in K} \left|g(s) - f(s)\right| < \varepsilon\right\}.$$

The boundaries

$$\left\{g \in H(D)\colon \sup_{s\in K} \left|g(s) - f(s)\right| = \varepsilon\right\}$$

of $A(f, \varepsilon, K)$ do not intersect for fixed $f(s)$ and K. Therefore they have P_ζ-measure zero for all $\varepsilon > 0$ except maybe for a finite or countable set. Hence the sets $A(f, \varepsilon, K)$ are the continuity sets of the measure P_ζ for all $\varepsilon > 0$ except maybe for a finite or countable set. It is well known, see, for example [2], that P_n converges weakly to P as $n \to \infty$ if and only if

$$\lim_{n\to\infty} P_n(A) = P(A)$$

for all continuity sets of the measure P.

Now let $\varepsilon > 0$ be such that a set $A(f, \varepsilon, K)$ is a continuity set of the measure P_ζ. Then the above remark and Theorems 2 and 3 yield

$$\lim_{T\to\infty} \nu_T\big(\zeta(s+i\tau) \in A(f, \varepsilon, K)\big) = P_\zeta\big(A(f, \varepsilon, K)\big).$$

Consequently,

$$\nu_T\big(\zeta(s+i\tau) \in A(f, \varepsilon, K)\big) = P_\zeta\big(A(f, \varepsilon, K)\big) + R_T(f, \varepsilon, K) \tag{1}$$

where

$$\lim_{T\to\infty} R_T(f, \varepsilon, K) = 0 .$$

It follows from Theorem 4 and equality (1) that if $f(s)$ and K satisfy the conditions of Theorem 4 then

$$P_\zeta\big(A(f, \varepsilon, K)\big) > 0 . \tag{2}$$

We recall that by the definition of the measure P_ζ

$$P_\zeta\big(A(f, \varepsilon, K)\big) = m\big(\omega \in \Omega\colon\ \zeta(s, \omega) \in A(f, \varepsilon, K)\big) \stackrel{def}{=} H(f, \varepsilon, K).$$

Moreover

$$H(f, \varepsilon, K) = \prod_p m_p\big(\omega\colon\ \omega(p) \in A_p(f, \varepsilon, K)\big)\,,$$

where

$$A_p(f, \varepsilon, K) = \Big\{\omega(p):\ \sup_{s\in K}\big|\zeta(s, \omega) - f(s)\big| < \varepsilon\Big\}\,,$$

and m_p denotes the Haar measure on $\big(\gamma_p, \mathcal{B}(\gamma_p)\big)$.

Now we state a theorem on the effectivization of the universality theorem for the Riemann zeta-function.

THEOREM 5. *Let $f(s)$ and K be defined in Theorem 4, and let $\varepsilon > 0$ be such that the set $A(f, \varepsilon, K)$ is a continuity set of the measure P_ζ. Suppose that T satisfies the inequality*

$$\big|R_T(f, \varepsilon, K)\big| < H(f, \varepsilon, K)\,.$$

Then there exists a real number $\tau \in [0, T]$ such that

$$\sup_{s\in K}\big|\zeta(s + i\tau) - f(s)\big| < \varepsilon\,.$$

Proof follows immediately from (1) and (2) and from the definition of $H(f, \varepsilon, K)$.

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Pastaba apie Rymano dzeta funkcijos universalumą

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Straipsnyje nagrinėjama universalumo teoremos Rymano dzeta funkcijai efektyvizacijos problema.