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On the Euler function

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Let $\varphi(n)$ denote the function of Euler, i.e. the number of positive integers not exceeding n, which are relatively prime to n. The miscellaneous results on this function are known.

We consider the sum

$$A(x)=\sum_{\varphi(n)\leqslant x} 1.$$

In this paper we survey the results on asymptotic behaviour of the sum $A(x)$.

The proof of the equality

$$\lim_{x\to\infty}\frac{A(x)}{x}=a \tag{1}$$

with

$$a=\frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

where $\zeta(s)$ is the Riemann zeta-function, is given by P. Erdös in 1945 [7]. His proof is based on the theorem of Schoenberg [10], that $n/\varphi(n)$ possesses a distribution function. The other proof of (1) was given by R. Dressler [6] and also the equality (1) follows from the Wiener–Ikehara theorem [4].

To obtain the estimate of the remainder term

$$R(x)=A(x)-ax,$$

the consideration of generating function

$$F(s)=\sum_{n=1}^{\infty}\frac{1}{\varphi(n)^s}=\zeta(s)G(s),\quad s=\sigma+it,\ \sigma>1,$$

where

$$G(s)=\prod_p\left(1+\frac{1}{(p-1)^s}-\frac{1}{p^s}\right),$$

is necessary. On the other hand the generating function $F(s)$ is closely connected with the zeta function $Z(s)$ of corresponding set of generalized integers. If more exactly, let

$$Z(s)=\prod_{p>2}\left(1-\frac{1}{(p-1)^s}\right)^{-1}=\sum_{i=1}^{\infty}\beta_i\nu_i^{-s},\quad \sigma>1,$$

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where $\nu_1 = 1, \nu_2, \nu_3, \ldots$ is increasing sequence of generalized integers generated by sequence of generalized primes $\{p-1\}$ (p are rational prime numbers), $\beta_1 = 1, \beta_2, \beta_3, \ldots,$ are non-negative integers. Then

$$F(s) = Z(s)H(s),$$

where $H(s)$ is analytic in the half–plane $\sigma > \frac{1}{2}$. Besides,

$$H(s) = \sum_{k=1}^{\infty} d_k k^{-s} = \prod_{p>2}\Big(1 - \big((p-1)^{-2s} - (p^2-p)^{-s}\big)\sum_{j=0}^{\infty} p^{-js}\Big).$$

If $B(x)$ is the number of generalized integers ν_i not exceeding x, i.e.

$$B(x) = \sum_{\nu_i \leqslant x} \beta_i,$$

then [4]

$$A(x) = \sum_{k=1}^{[x]} d_k B\Big(\frac{x}{k}\Big).$$

Due to this equality the asymptotic formula for $A(x)$ may be deduced from the asymptotics of $B(x)$. In this way in 1972 P. Bateman [4] deduce the estimate

$$R(x) = O\big(x \exp\{-c(\log x)^{1/3}\}\big)$$

for any $c > 0$ from theorems of Beurling's generalized numbers theory. In the same paper the classical method of contour integration gives more exact result

$$R(x) = O_c\big(x \exp\{-c(\log x \log\log x)^{1/2}\}\big) \tag{2}$$

for every constant $c < 1/\sqrt{2}$. The proof of P. Bateman uses such evaluation of $|G(s)|$ in the strip $0 < \sigma < 1$: if $|t| \geqslant 8$ and $\sigma \geqslant \sigma_0(t)$, $\sigma_0(t)$ is some function such that $\frac{1}{3} \leqslant \sigma_0(t) \leqslant 1$, then

$$|G(s)| \leqslant \exp\big\{50|t|^{1-\sigma_0(t)} \log\log|t|\big\}. \tag{3}$$

J.-L. Nicolas [9], A. Smati [11], M. Balazard and A. Smati [2] have studied the elementary methods of estimation of $R(x)$. But their methods give the estimates, which are weaker than (2).

Only in 1996 [12] the method of P. Bateman gives the estimate

$$R(x) = O\bigg(x \exp\bigg\{-\Big(1 + \frac{c\log\log\log x}{\log\log x}\Big)\Big(\frac{1}{2}\log x \log\log x\Big)^{1/2}\bigg\}\bigg)$$

for some constant $c > 0$. Thus the estimate (2) is valid for $c \leqslant \frac{1}{\sqrt{2}}$. In the proof the slightly more exact evaluation of $|G(s)|$ is used.

In the same 1996 year M. Balazard [1] proved the formula (2) with $c < \sqrt{2}$. His proof is elementary based on the application of H. Diamond's [5] theorem.

Also in 1996 M. Balazard and G. Tenenbaum [3] proved the best result up to date: for some constant $c > 0$ the estimate

$$R(x) = O\big(x \exp\big\{ -c(\log)^{3/5}(\log\log x)^{-1/5}\big\}\big) \tag{4}$$

is true. The method of the proof is analytic and uses the more exact than (3) evaluation of $G(s)$: there exists the constant $k > 0$, for which

$$G(s) = O\big((\log T)^{4/3}(\log\log T)^{2/3}\big)$$

for $\sigma \geqslant 1 - k\beta(T)$, $T = |t| + 3$, where

$$\beta(T) = (\log T)^{-2/3}(\log\log T)^{-1/3}.$$

The proof of evaluation (4) is based on the results of Karatsuba [8], i.e. it is necessary to estimate the trigonometric sums

$$S_N = \sum_{M<p\leqslant N} (p-1)^{it}, \quad M < N \leqslant 2M.$$

Probably the error $R(x)$ may be evaluated more exactly than (4). P. Erdös conjecture is

$$R(x) = O\big(x \exp\big\{ -(\log x)^{1-\varepsilon}\big\}\big)$$

for every $\varepsilon > 0$.

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Apie Oilerio funkciją

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Tegu $\varphi(n)$ – Oilerio funkcija. Nagrinėsime sumą

$$A(x) = \sum_{\varphi(n)\leqslant x} 1.$$

Yra žinoma, kad

$$\lim_{x\to\infty} \frac{A(x)}{x} = a, \quad a = \frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

kur $\zeta(s)$ – Rymano dzeta funkcija. Straipsnyje apžvelgiami metodai, leidžiantys įvertinti asimptotinės formulės

$$A(x) = ax + R(x), \quad x \to \infty$$

liekamąjį narį $R(x)$ bei supažindinama su naujausiais autoriaus rezultatais.