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Prefixed tableaus for three-valued modal propositional logics

J. Sakalauskaitė (MII)

1. INTRODUCTION

In this paper we investigate tableau-based theorem proving for some of the three-valued modal propositional logics. We consider three-valued counterparts (two for each logic) of two-valued modal logics K, $K4$, T, B, $S4$, $S5$. We denote them by K_i, $K4_i$, T_i, B_i, $S4_i$, $S5_i, i \in \{2,3\}$. A notion of Kripke frame for these logics is as in two-valued case, but now truth values at possible worlds are from the set $\{true, false, undefined\}$. These counterparts (except for logics $K4$ *and* B) are considered by Takano in [3], where the cut free sequent calculi for these logics are constructed.

In this paper we introduce the prefixed tableau systems for the logics K_i, $K4_i$, T_i, B_i, $S4_i$, $S5_i$, $i \in \{2,3\}$ and prove completeness theorem for these systems. Prefixed tableau systems for two-valued modal case was elaborated by Fitting in [2]. In a prefixed tableau each formula has a prefix. Prefixes of formulas represent the names of possible worlds and the accessibility relation between worlds is reflected by syntactic features of these names.

The paper is organized as follows. In section 2 we introduce the syntax and semantics of the three-valued modal logics. In the next section we present the prefixed tableau systems for these logics and prove soundness and completeness of the systems.

2. SYNTAX AND SEMANTICS

We put $T = \{1, 2, 3\}$ and will use T as the set of *truth values*. Intuitively, the truth values 1, 2, 3 stand for 'true', 'undefined' and 'false', respectively. We let $\lambda, \mu, \nu, \ldots$ denote truth values.

Formulas are constructed from propositional variables by means of propositional connectives and the necessity operator $\Box$; we assume for each propositional connective F, the arity $\alpha(F) \geq 0$ and the truth function $f_F : T^{\alpha(F)} \to T$ are predetermined.

Definition 2.1. *A valued formula* is any pair of a formula and a truth value.

Definition 2.2. *A (three-valued) Kripke frame* is the triplet (W, R, v), where W is a nonempty set (set of worlds), R is a binary relation on W, v is a mapping which assigns a truth value from T to each pair of propositional variable and an element of W.

Definition 2.3. Suppose that (W, R, v) is a Kripke frame and $i \in \{2, 3\}$. We call the triplet (W, R, v^i) a (three-valued) Kripke structure of type i (generated from (W, R, v)), if v^i is the mapping which assigns a truth value to each pair of a formula and an element of W and is defined by recursion as follows:

$v^i(p, s) = v(p, s)$, where p is a propositional variable;

$$v^i(F(A_1, \ldots, A_{\alpha(F)}), s) = f_F(v^i(A_1, s), \ldots, v^i(A_{\alpha(F)}, s));$$

$$v^2(\Box A, s) = \begin{cases} 1, & \text{if } sRt \text{ implies } v^2(A, t) = 1 \text{ for every } t \in W; \\ 2, & \text{if } sRt \text{ and } v^2(A, t) = 2 \text{ for some } t \in W; \\ 3, & \text{otherwise.} \end{cases}$$

$$v^3(\Box A, s) = \begin{cases} 1, & \text{if } sRt \text{ implies } v^3(A, t) = 1 \text{ for every } t \in W; \\ 3, & \text{otherwise.} \end{cases}$$

Let L be a modal logic. Models of L are defined as follows.

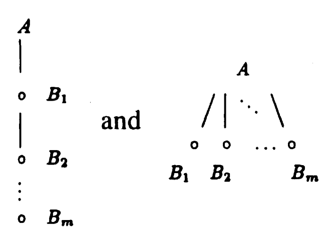
Definition 2.4. Models of K_i are nothing but the Kripke structures of type i; whereas a model of T_i, $K4_i$, B_i, $S4_i$ is a Kripke structure (W, R, v^i) of type i such that R is reflexive, transitive, reflexive and symmetric, reflexive and transitive, respectively. A model of $S5_i$ is a Kripke structure (W, R, v^i), where R is reflexive, symmetric and transitive.

3. PREFIXED PROPOSITIONAL TABLEAUS

We shall consider informally the notions of trees, branches, nodes, etc. We consider the following symbols:

$$\frac{A}{B_1 \circ B_2 \circ \ldots \circ B_m} \quad \text{and} \quad \frac{A}{B_1 + B_2 + \ldots + B_m}$$

as denoting trees of the following form, respectively:



and we shall abbreviate those symbols by the following expressions:

$$\frac{A}{\circ\{B_i : i \leq m\}} \qquad \frac{A}{+\{B_i : i \leq m\}}$$

Definition 3.1. A *prefix* is a finite sequence of positive integers. A *prefixed valued formula* $\sigma(A, \lambda)$ is a prefix σ followed by a valued formula (A, λ).

We will systematically use σ, σ', etc. for prefixes throughout this paper.

The idea is, we will interpret prefixes as naming worlds in some model. $\sigma(A, \lambda)$ means that under this model A is forced to have value λ in the world σ names.

We define tableau rules as follows.

These rules are divided in two parts: the rules for formulas of the form $F(A_1, \ldots, A_m)$, where F is a propositional connective, and the rules for modalized formulas. We begin with formulas of the form $F(A_1, \ldots, A_m)$. We present the rules for these formulas following [1]. (In fact, these rules are obtained from the definition of the rules for the formulas of the form $F(A_1, \ldots, A_m)$ from [1] by omitting the first condition.)

For each valued prefixed formula $\sigma(F(A_1, \ldots, A_m), \lambda)$, where F is an m-ary propositional connective, we define the rule as follows:

$$\frac{\sigma\big(F(A_1, \ldots, A_m), \lambda\big)}{\begin{array}{c}+\{\sigma(A_{i_1}, \lambda_{j_1})\bigcirc\ldots\bigcirc\sigma(A_{i_t}, \lambda_{j_t}) : \quad \lambda_{j_1}, \ldots, \lambda_{j_t} \leq 3,\ t \leq m \\ \text{and the propositional condition } H_\lambda(F; \lambda_{j_1}, \ldots, \lambda_{j_t}) \text{ holds}\}\end{array}},$$

where $H_\lambda(F; \lambda_{j_1}, \ldots, \lambda_{j_t})$ means that

1) if f represents the connective F, then

$$f(\nu_1, \ldots, \nu_{i_1}, \ldots, \nu_{i_2}, \ldots, \nu_{i_t}, \ldots, \nu_m) = \lambda$$

for all values of the function f, where $\nu_{i_k} = \lambda_{j_k}$ and the other $\nu's$ are arbitrary; and

2) no $t' < t$ satisfies 1).

Before presenting rules for modalized formulas we present a little more terminology. The following two definitions are borrowed from [2].

Definition 3.2. We say a prefix σ is *used* on a tableau branch if σZ occurs on the branch for some valued formula Z. We say a prefix σ is *unrestricted* on a tableau branch if σ is not an initial segment (proper or otherwise) of any prefix used on the branch.

Let σ be an arbitrary prefix.

Definition 3.3. We say the relation of *accessibility from* on prefixes satisfies:

1) the *general* condition if σ, n is accessible from σ for every integer n; 2) the *reverse* condition if σ is accessible from σ, n for every integer n; 3) the *identity* condition if σ is accessible from σ; 4) the *transitivity* condition if the sequence σ, σ' is accessible from σ for every non-empty sequence σ'.

Now, for the various logics we consider, the accessibility relation on prefixes is given in the following chart.

For the logic	the accessibility relation on prefixes meets the condition:
K_i	general
$K4_i$	general, transitivity
T_i	general, identity
B_i	general, identity, reverse

$S4_i$	general, identity, transitivity
$S5_i$	no special conditions, any prefix is accessible from any other

Now we present the tableau rules for modalized formulas. For the logics with $i = 2$ these rules are defined as follows.

$\frac{\sigma(\Box A, 1)}{\sigma'(A, 1)}$, where σ' has been used on the branch and is accessible from σ.

$\frac{\sigma(\Box A, 2)}{\sigma, n(A, 2)}$, where σ, n is unrestricted prefix.

$\frac{\sigma(\Box A, 3)}{\sigma'(A, 1) + \sigma'(A, 3)}$, where σ' has been used on the branch and is accessible from σ.

$\frac{\sigma(\Box A, 3)}{\sigma, n(A, 3)}$, where σ, n is urestricted prefix.

For the logics with $i = 3$ the tableaux rules for modalized formulas are defined as follows.

$\frac{\sigma(\Box A, 1)}{\sigma'(A, 1)}$, where σ' has been used on the branch and is accessible from σ.

$\frac{\sigma(\Box A, 3)}{\sigma, n(A, 2) + \sigma, n(A, 3)}$, where σ, n is unrestricted prefix.

Definition 3.4. A prefixed formula $\sigma(A, \lambda)$ which occurs over the line of a rule is called a *premise* of the rule.

Definition 3.5. Let $+\{\sigma(A, \lambda_1^j)\circ \dots \circ\sigma(A, \lambda_{n_j}^j),\ j \leq K\}$, $K \geq 1$ be the expression below the line in a tableau rule. We say that $\sigma(A, \lambda_1^j)\circ \dots \circ\sigma(A, \lambda_{n_j}^j)$ is a *consequence* of this rule.

A tableau is a tree, with each node labelled with a prefixed formula.

Definition 3.6. *A tableau* for a formula (A, λ) is any tree whose first node is the formula $1(A, \lambda)$ and those next nodes are determined by the following procedure: if a branch of the tree contains a prefixed formula σZ and a tableau rule with the premise σZ is defined, then this branch can be extended by adding new nodes through the application of this rule to σZ (following the convention that formulas separated by "$\circ$" go in the same branch and sets of formulas separated by "+" go into different branches).

Definition 3.7. A tableau branch is *closed* if it contains both $\sigma(A, \lambda)$ and $\sigma(A, \mu)$, $\lambda \neq \mu$ or the branch contains some nonatomic prefixed formula $\sigma(A, \lambda)$ and there exists no defined rule with the premise $\sigma(A, \lambda)$. A tableau is *closed* if each branch of it is closed.

We refer to the accessibility notion on prefixes that is appropriate for a logic L as L-accessibility for short.

Let L_i be a logic we are considering ,i.e. $i \in \{2, 3\}$, $L \in \{K, K4, T, B, S4, S5\}$. Let S be a set of prefixed formulas and let $\mathcal{M} = (W, R, v^i)$ be a L_i-model.

Definition 3.8. By an *L_i-interpretation* of S in the model $\mathcal{M}$ we mean a mapping I from the set of prefixes that occur in S to W such that if a prefix τ is L_i-accessible from a prefix σ, then $I(\sigma)RI(\tau)$. S is *L_i-satisfiable under the L_i-interpretation I* if for each $\sigma(A, \lambda) \in S$ $v^i(A, I(\sigma)) = \lambda$. S is *L_i- satisfiable* if S is L_i-satisfiable under some L_i-interpretation.

Loosely, a set of prefixed formulas is L_i-satisfiable if it partially describes some model.

Definition 3.9. A tableau is *L_i-satisfiable* if some branch of it is L_i-satisfiable. A branch is *L_i-satisfiable* if the set of prefixed formulas on it is L_i-satisfiable.

Let L_i be a logic as above.

LEMMA 3.1. *Suppose $\mathcal{T}$ is a prefixed tableau that is L_i-satisfiable. Let $\mathcal{T}'$ be the tableau that results from a single L_i-tableau rule being applied to $\mathcal{T}$. Then $\mathcal{T}'$ is also L_i-satisfiable.*

Proof as the proof of the lemma 3.1 in the chapter eight in [2].

COROLLARY 3.2 (soundness). *If there exists a closed tableau for* $1(A, \lambda)$ *then for each L_i-model (W, R, v^i) and each world $s \in W$ $v^i(A, s) \neq \lambda$.*

Proof. Suppose there exists a closed tableau for $1(A, \lambda)$, but there is L_i-model (W, R, v^i) and a world $s \in W$ such that $v^i(A, s) = \lambda$. Define a L_i-interpretation I by setting $I(1) = s$. It follows that the starting L_i-tableau $\{1(A, \lambda)\}$ is L_i-satisfiable. Then by lemma 3.1, so is every subsequent L_i-tableau. But an L_i-satisfiable tableau can not be closed, contradicting the assumption.

Now for every logic L we are considering we prove the completeness theorem.

Let L_i be a logic we are considering, i.e. $i \in \{2, 3\}$, $L \in \{K, K4, T, B, S4, S5\}$.

THEOREM 3.3 (completeness). *If in each L_i-model (W, R, v^i) and each world $s \in W$ $v^i(A, s) \neq \lambda$, then there exists a closed tableau with the root* $1(A, \lambda)$.

Proof of this theorem follows the lines of the proof of completeness for prefixed tableaus for two-valued modal logics presented in [2]. We omit the proof here.

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Prefiksinės lentelės trireikšmėms modalinėms propozicinėms logikoms

J. Sakalauskaitė

Pateiktos lentelių sistemos trireikšmėms modalinėms propozicinėms logikoms. Šioms sistemoms įrodytos neprieštaringumo ir pilnumo teoremos.