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Pipelined-block discrete-time system modeling in state space

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Consider a multi-input, multi-output LTV discrete-time system described by dynamic equations

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{1a}$$

$$y(k) = C(k)x(k) + D(k)u(k), \quad k = 0, 1, 2, \dots \tag{1b}$$

where the state $x(k)$ is $N \times 1$, the state update matrix $A(k)$ is $N \times N$, $B(k)$ is $N \times P$, $C(k)$ is $R \times N$, $D(k)$ is $R \times P$, $u(k)$ is $P \times 1$, and $y(k)$ is $R \times 1$.

The general solution of the dynamic equation (1a) is given by [1]

$$x(n) = F(n,k)x(k) + \sum_{j=k}^{n-1} F(n, j+1)B(j)u(j), \quad n = 0, 1, 2, \dots, \tag{2}$$

where $F(n,n) = I_N$,

$$F(n,k) = \prod_{i=1}^{n-k} A(n-i). \tag{3}$$

Substituting $x(n)$ from (2) into (1b), we obtain

$$y(n) = C(n)F(n,k)x(k) + \sum_{j=k}^{n-1} C(n)F(n, j+1)B(j)u(j) + D(n)u(n). \tag{4}$$

Model 1. Substituting $n = kM + M$, and $k = kM$ into (2), where M is pipelining level, we have

$$\begin{aligned} x(kM+M) &= F(kM+M, kM)x(kM) + \sum_{j=kM}^{kM+M-1} F(kM+M, j+1)B(j)u(j) \\ &= F(kM+M, kM)x(kM) \\ &\quad + \sum_{j=1}^{M} F(kM+M, kM+j)B(kM+j-1)u(kM+j-1). \end{aligned} \tag{5}$$

We obtain from (3) and (5) the state equation of the pipelined-block LTV discrete-time system in matrix form:

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k), \qquad k = 0, 1, 2, \dots, \tag{6}$$

where the $N \times N$ matrix $\bar{A}(k)$ is defined by

$$\bar{A}(k) = \prod_{i=1}^{M} A(kM + M - i). \tag{7}$$

The $N \times MP$ matrix $\bar{B}(k)$ is defined by

$$\bar{B}(k) = \left[B_1, \ldots, B_j, \ldots, B_M\right], \tag{8}$$

in which

$$B_j = \prod_{i=1}^{M-j} A(kM + M - i)B(kM + j - 1), \qquad j = 1, 2, \ldots, M - 1,$$

$$B_M = B(kM + M - 1),$$

$$\bar{u}(k) = \left[u(kM), \ldots, u(kM + M - 1)\right]^T,$$

$$\bar{x}(k) = x(kM), \quad \bar{x}(k+1) = x\left[(k+1)M\right].$$

Substituting $n = kM + i - 1$, $k = kM$, $k = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, M$ into (4), we obtain

$$y(kM + i - 1) = C(kM + i - 1)F(kM + i - 1, kM)x(kM) + \sum_{j=1}^{i-1} C(kM + i - 1)F(kM + i - 1, kM + j)B(kM + j - 1)u(kM + j - 1) + D(kM + i - 1)u(kM + i - 1). \tag{9}$$

Then we get from (3) and (9) the output equation of the pipelined-block LTV discrete-time system in matrix form:

$$\bar{y}(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k), \qquad k = 0, 1, 2, \ldots, \tag{10}$$

where the $MR \times N$ matrix $\bar{C}(k)$ is defined by

$$\bar{C}(k) = \left[C_1, \ldots, C_i, \ldots, C_M\right]^T, \tag{11}$$

in which

$$C_1 = C(kM), \qquad C_i = C(kM + i - 1)\prod_{j=2}^{i} A(kM + i - j), \qquad i = 2, 3, \ldots, M.$$

The $MR \times MP$ matrix $\bar{D}(k)$ is defined by $R \times P$ matrices D_{ij}

$$\bar{D}(k) = \left[D_{ij}\right], \quad i, j = 1, 2, \ldots, M, \tag{12}$$

in which

$$D_{ij} = \mathbf{0}, \quad \text{if } i < j,$$
$$D_{ij} = D(kM + i - 1), \quad \text{if } i = j,$$
$$D_{ij} = C(kM + i - 1)B(kM + j - 1), \quad \text{if } i = j + 1,$$
$$D_{ij} = C(kM + i - 1)\prod_{l=2}^{i-j} A(kM + i - l)B(kM + j - 1), \quad \text{if } i > j + 1;$$
$$\bar{y}(k) = \left[y(kM), \ldots, y(kM + M - 1)\right]^T.$$

For *LPTV discrete-time systems* $A(k)$, $B(k)$, $C(k)$, and $D(k)$ are L-periodic, i.e., $A(k + L) = A(k)$, $B(k + L) = B(k)$, $C(k + L) = C(k)$, and $D(k + L) = D(k)$. In case the pipelining level M is equal to the periodicity L of the LPTV system, we obtain from (7), (8), (11), and (12) simpler matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$. For example, we have from (7)

$$\bar{A}(k) = \bar{A} = \prod_{i=1}^{L} A(L - i).$$

For *LTI discrete-time systems* $A(k) = A$, $B(k) = B$, $C(k) = C$, and $D(k) = D$. We obtain from (7), (8), (11), and (12) matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. For example, we have from (7) $\bar{A}(k) = \bar{A} = A^M$.

In the pipelined-block discrete-time system, we use $\bar{x}(0)$ to compute the block of outputs $y(0), \ldots, y(M - 1)$, and to update $\bar{x}(1) = x(M)$. In the next cycle, $\bar{x}(1)$ is used to compute the next block of outputs, and to update the state $\bar{x}(2) = x(2M)$, an so on.

Model 2. Substituting $n = k + M$ into (2), where M is a pipelining level, $k = 0, 1, 2, \ldots$, we get

$$x(k + M) = F(k + M, k)x(k) + \sum_{j=1}^{M} F(k + M, k + j)B(k + j - 1)u(k + j - 1).$$

where

$$F(k + M, k) = \prod_{i=1}^{M} A(k + M - i), \qquad F(k + M, k + j) = \prod_{i=1}^{M-j} A(k + M - i),$$

or in a matrix form:

$$x(k + M) = \bar{A}(k)x(k) + \bar{B}(k)\bar{u}(k), \qquad k = 0, 1, 2, \ldots, \tag{13}$$

where the $N \times N$ matrix

$$\bar{A}(k) = \prod_{i=1}^{M} A(k + M - i). \tag{14}$$

The $N \times MP$ matrix $\bar{B}(k)$ is defined by

$$\bar{B}(k) = \left[B_1, \ldots, B_j, \ldots, B_M\right], \tag{15}$$

in which

$$B_j = \prod_{i=1}^{M-j} A(k+M-i)B(k+j-1), \quad j = 1, 2, \ldots, M-1, \quad B_M = B(k+M-1).$$

and

$$\bar{u}(k) = \left[u(k), u(k+1), \ldots, u(k+M-1)\right]^T.$$

Assume $k = mM + l$, $m = 0, 1, 2, \ldots$, $l = 0, 1, \ldots, M-1$. From (13) and (1b). we obtain Model 2 of the pipelined-block LTV discrete-time system:

$$x(mM + M + l) = \bar{A}(mM + l)x(mM + l) + \bar{B}(mM + l)\bar{u}(mM + l),$$
$$y(mM + l) = C(mM + l)x(mM + l) + D(mM + l)u(mM + l),$$

or in a matrix form:

$$X(m+1) = \bar{A}(m)X(m) + \bar{B}(m)\bar{U}(m), \tag{16a}$$
$$Y(m) = \bar{C}(m)X(m) + \bar{D}(m)U(m), \quad m = 0, 1, 2, \ldots, \tag{16b}$$

where $\bar{A}(m)$, $\bar{B}(m)$, $\bar{C}(m)$, and $\bar{D}(m)$ are $NM \times NM$, $NM \times M^2P$, $RM \times NM$, $RM \times PM$ diagonal matrices, respectively, defined by

$$\bar{A}(m) = \text{diag}\left\{\bar{A}(mM), \ldots, \bar{A}(mM + l), \ldots, \bar{A}(mM + M - 1)\right\},$$
$$\bar{B}(m) = \text{diag}\left\{\bar{B}(mM), \ldots, \bar{B}(mM + l), \ldots, \bar{B}(mM + M - 1)\right\},$$
$$\bar{C}(m) = \text{diag}\left\{\bar{C}(mM), \ldots, \bar{C}(mM + l), \ldots, \bar{C}(mM + M - 1)\right\},$$
$$\bar{D}(m) = \text{diag}\left\{\bar{D}(mM), \ldots, \bar{D}(mM + l), \ldots, \bar{D}(mM + M - 1)\right\},$$

in which

$$\bar{A}(mM + l) = \prod_{i=1}^{M} A(mM + l + M - i), \qquad \bar{B}(mM + l) = \left[B_1, \ldots, B_j, \ldots B_M\right].$$

where

$$B_j = \prod_{i=1}^{M-j} A(mM + M + l - i)B(mM + l + j - 1), \quad j = 1, 2, \ldots, M-1.$$
$$B_M = B(mM + l + M - 1);$$

$X(m+1)$ and $X(m)$ are $NM \times 1$ vectors given by

$$X(m+1) = \left[x(mM + M), \ldots, x(mM + 2M - 1)\right]^T,$$

$$X(m) = \left[x(mM), \ldots, x(mM + M - 1)\right]^T,$$

$\bar{U}(m)$ is $M^2P \times 1$ vector given by $\bar{U}(m) = [\bar{u}(mM), \ldots, \bar{u}(mM+M-1)]^T$, $Y(m)$ and $U(m)$ are $RM \times 1$ and $PM \times 1$ vectors, respectively, given by

$$Y(m) = \left[y(mM), \ldots, y(mM + M - 1)\right]^T,$$

$$U(m) = \left[u(mM), \ldots, u(mM + M - 1)\right]^T.$$

For *LPTV systems*, if M is equal to periodicity, then

$$\bar{A}(mM + l) = \bar{A}(l) = \prod_{i=1}^{M} A(l + M - i),$$

$$\bar{B}(mM + l) = \bar{B}(l) = \left[B_1, \ldots, B_j, \ldots, B_M\right],$$

in which

$$B_j = \prod_{i=1}^{M-j} A(l+M-i)B(l+j-1), \quad j = 1, 2, \ldots, M-1, \qquad B_M = B(l+M-1);$$

$C(mM + l) = C(l)$ and $D(mM + l) = D(l)$, $l = 0, 1, \ldots, M - 1$.

For *LTI systems*,

$$\bar{A}(mM + l) = \bar{A} = A^M, \qquad \bar{B}(mM + l) = \bar{B} = \left[A^{M-1}B, \ldots, B\right],$$
$$C(mM + l) = C, \qquad D(mM + l) = D, \quad l = 0, 1, \ldots, M - 1.$$

Model 3. Substituting $n = kM + i$, and $k = kM$, $n, k = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, M$ into (2), we get

$$x(kM + i) = F(kM + i, kM)x(kM) + \sum_{j=1}^{i} F(kM + i, kM + j)B(kM + j - 1)u(kM + j - 1). \tag{17}$$

Then we obtain from (3) and (17) the state equation of the pipelined-block LTV discrete-time system:

$$x(kM + i) = \prod_{j=1}^{i} A(kM + i - j)x(kM) + \sum_{j=1}^{i}\prod_{l=1}^{i-j} A(kM + i - l)B(kM + j - 1)u(kM + j - 1),$$

or in matrix form:

$$X(k) = \bar{A}(k)x(kM) + \bar{B}(k)\bar{u}(k), \tag{18}$$

where $NM \times N$ matrix $\bar{A}(k) = [\bar{A}(k, 1), \dots, \bar{A}(k, i), \dots, \bar{A}(k, M)]^T$, in which

$$\bar{A}(k, i) = \prod_{j=1}^{i} A(kM + i - j), \quad i = 1, 2, \dots, M;$$

the $MN \times MP$ lower triangular block matrix $\bar{B}(k)$ is defined by

$$\bar{B}(k) = \left[\bar{B}_{ij}\right], \quad i, j = 1, 2, \dots, M,$$

in which matrix

$$\bar{B}_{ij} = \mathbf{0}, \quad \text{if } i < j,$$
$$\bar{B}_{ij} = B(kM + j - 1), \quad \text{if } i = j,$$
$$\bar{B}_{ij} = \prod_{l=1}^{i-j} A(kM + i - l)B(kM + j - 1), \quad \text{if } i > j;$$
$$X(k) = \left[x(kM + 1), \dots, x(kM + M)\right]^T,$$

and

$$\bar{u}(kM, i) = \left[u(kM), \dots, u(kM + M - 1)\right]^T,$$

REFERENCES

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Konvejerinių-blokinių diskretinių sistemų modeliavimas būsenų erdvėje

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Straipsnyje parodyta, kad tiesinių daugiamačių kintamų parametrų sistemų, tiesinių sistemų su periodiškai kintančiais parametrais ir tiesinių sistemų su pastoviais parametrais įvairūs modeliai būsenų erdvėje gali būti gauti iš bendrojo dinaminių lygčių sprendinio.