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On A-decomposition of probability measures in Hilbert spaces

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Let H be a real separable Hilbert space and

$$\xi_1, \xi_2, \ldots$$

be a sequence of independent H-valued random variables (i.H.r.v.), the following well-known result will be helpful.

KOLMOGOROV'S THREE–SERIES CRITERION. *In order that the series*

$$\sum_{n=1}^{\infty} \xi_n \tag{1}$$

converge (weakly), it is necessary that for any $\varepsilon > 0$ the three series

$$\sum_{n=1}^{\infty} \int\limits_{\|x\|<\varepsilon} x \, d\nu_n(x), \tag{2}$$

$$\sum_{n=1}^{\infty} \int\limits_{\|x\|<\varepsilon} \|x - \int\limits_{\|y\|<\varepsilon} y \, d\nu_n(y)\| \, d\nu_n(x) \tag{3}$$

and

$$\sum_{n=1}^{\infty} \mathbf{P}\{\|\xi_n\| \geqslant \varepsilon\} \tag{4}$$

be convergent (here the probability measures ν_n are the distributions of random variables ξ_n correspondingly), and it is sufficient that the series converge for at least one $\varepsilon > 0$.

We call a linear operator A an a-operator if for all $x \in H$ $\|Ax\| = a\|x\|$. First we prove a following result:

THEOREM 1. *Let $\{\xi_n\}$ be a sequence of i.H.r.v. and let $\xi_n \overset{L}{=} A_n\xi$ („$\overset{L}{=}$" means that both sides have the same distribution), where ξ is H-valued random variable (H.r.v.), A_n are a_n-operators with the properties*:

i) $a_n > 0, \ n = 1, 2, \ldots;$

ii) *for some N $a_{n+1} \leqslant a_n$, when $n \geqslant N$;*

iii) $\sum_{n=1}^{\infty} a_n < \infty$.

In order that the series (1) *converge it is necessary that for any* $\varepsilon > 0$ *the series*

$$\sum_{j=N+1}^{\infty} j\mathbf{P}\{\varepsilon a_{j-1}^{-1} \leqslant \|\xi\| < \varepsilon a_j^{-1}\} \tag{5}$$

be convergent, and sufficient that the series (5) *converge for at least one* $\varepsilon > 0$.

Proof. Necessity. If the series (1) converges, then also the series (4) converges and we have

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}\{\|\xi_n\| > \varepsilon\} &= \sum_{n=1}^{\infty} \mathbf{P}\{\|\xi\| > \varepsilon a_n^{-1}\}\\ &= \sum_{n=1}^{N} \mathbf{P}\{\|\xi\| > \varepsilon a_n^{-1}\} + \sum_{n=N+1}^{\infty} \sum_{j=n+1}^{\infty} \mathbf{P}\{\varepsilon a_{j-1}^{-1} < \|\xi\| \leqslant \varepsilon a_j^{-1}\}\\ &= \sum_{n=1}^{\infty} \mathbf{P}\{\|\xi\| > \varepsilon a_n^{-1}\} + \sum_{j=N+1}^{\infty} j\mathbf{P}\{\varepsilon a_{j-1}^{-1} < \|\xi\| \leqslant \varepsilon a_j^{-1}\}\\ &\quad - N\mathbf{P}\{\|\xi\| > \varepsilon a_N^{-1}\}.\end{aligned} \tag{6}$$

Sufficiency. Let the series (5) be convergent. By the equations (6) we have that the series (4) is also convergent. We can prove convergence of the series (2) as follows:

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} \int_{\|x\|<1} x \, d\nu_n(x)\right\| &\leqslant \sum_{n=1}^{\infty} \int_{\|x\|<1} \|x\| \, d\nu_n(x)\\ &= \sum_{n=1}^{N} a_n \int_{\|y\|<a_n^{-1}} \|y\| \, d\nu(y) + \sum_{n=N+1}^{\infty} a_n \int_{\|y\|<a_N^{-1}} \|y\| \, d\nu(y)\\ &\quad + \sum_{n=N+1}^{\infty} a_n \sum_{j=N+1}^{n} \int_{a_{j-1}^{-1} \leqslant \|y\| < a_j^{-1}} \|y\| \, d\nu(y)\\ &= C_1 + \sum_{j=N+1}^{\infty} \sum_{n=j}^{\infty} a_n \int_{a_j^{-1} \leqslant \|y\| < a_j^{-1}} \|y\| d\nu(y)\\ &\leqslant C_1 + C_2 \sum_{j=N+1}^{\infty} n\mathbf{P}\{a_{j-1}^{-1} \leqslant \|\xi\| < a_j^{-1}\},\end{aligned}$$

here C_1 and C_2 are constants and probability measure ν is the distribution of the H.r.v. ξ.

Because of

$$\int_{\|x\|<1} \left\| x - \int_{\|y\|<1} y \, d\nu_n(y) \right\| d\nu_n(x) \leqslant 2 \int_{\|x\|<1} \|x\| \, d\nu_n(x)$$

we have also convergence of the series (3). This completes the proof.

Because of the inequalities

$$\sum_{j=1}^{\infty} j\mathbf{P}\{\varepsilon(j-1)^{\alpha} \leqslant \|\xi\| < \varepsilon j^{\alpha}\} - 1 \leqslant E\|\xi\|^{\frac{1}{\alpha}} \leqslant \sum_{j=1}^{\infty} j\mathbf{P}\{\varepsilon(j-1)^{\alpha} \leqslant \|\xi\| < \varepsilon j^{\alpha}\}$$

we have this corollary:

COROLLARY 1. *Let* $\{\xi_n\}$ *be a sequence of i.H.r.v, and let* $\xi_n \stackrel{L}{=} A_n \xi$, *where operators* A_n *are the* $n^{-\alpha}$*-operators for* $\alpha > 1$. *Then the series* (1) *converges if and only if* $E\|\xi\|^{\frac{1}{\alpha}} < \infty$.

If a probability measure μ is the distribution of a H.r.v. η and A is an invertible linear operator, by $A\mu$ we denote the distribution of the H.r.v. $A\eta$. If η and ξ are i.H.r.v. and for some linear operator A we have the equation

$$\eta \stackrel{L}{=} A\eta + \xi,$$

we have the decomposition

$$\mu = A\mu * \nu \tag{7}$$

of the probability measure μ. In this case we will say that probability measure μ is A-decomposable.

THEOREM 2. *Let* A *be a-operator with* $0 < a < 1$. *A probability measure* ν *can be a component in* A*-decomposition* (7) *of some probability measure* μ *if and only if*

$$\int_{\|x\|>1} \log \|x\| \, d\nu(x) < \infty. \tag{8}$$

Proof. It is easy to see that the equation (7) is equivalent to the equation

$$\eta \stackrel{L}{=} \sum_{n=1}^{\infty} \xi_n, \tag{9}$$

where $\{\xi_n\}$ is a sequence of i.H.r.v., $\xi_n \stackrel{L}{=} A^{n-1}\xi$ ($\xi_1 \stackrel{L}{=} \xi$). Because A^{n-1} are the a^{n-1}-operators, by Theorem 1 we have that the series (9) converges if and only if for $\varepsilon = 1$ converges the series (5).

Necessity of the condition (8) we can get as follows:

$$\int_{\|x\|>1} \log \|x\| \, d\nu(x) = \sum_{n=1}^{\infty} \int_{a^{-(n-1)} \leqslant \|x\| < a^{-n}} \log \|x\| \, d\nu(x)$$

$$\leqslant |\log a| \sum_{n=1}^{\infty} n\mathbf{P}\{a^{-(n-1)} \leqslant \|x\| < a^{-n}\}.$$

With the help of the following inequality we have *sufficiency*:

$$\int_{\|x\|>1} \log \|x\| \, d\nu(x) \geqslant |\log a| \left(\sum_{n=1}^{\infty} n\mathbf{P}\{a^{-(n-1)} \leqslant \|\xi\| < a^{-n}\} - 1 \right).$$

This completes the proof.

It follows that under condition (8) for all a-operators A with $0 < a < 1$ there exists a probability measure μ such that equation (7) is true.

Now we will consider only those probability measures ν in the decomposition (7) which are infinitely divisible. Our purpose is to describe them. We will show a more general result, from which the desirable result will follow.

As we know a H.r.v. ξ is infinitely divisible if and only if its characteristic functional (ch.f.) $f(y)$ is of the form

$$f(y) = \exp\left\{ i(x_0, y) - \frac{1}{2}(Sy, y) + \int \left[e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|^2} \right] dM(x) \right\}, \quad (10)$$

where (u, v), $u \in H$, $v \in H$ denote the inner product between u and v, x_0 is a fixed element of H, S is an S-operator and M is a σ-finite measure with finite mass outside every neighborhood of the origin and

$$\int_{\|x\|<1} \|x\|^2 \, dM(x) < \infty.$$

This representation is unique.

THEOREM 3. *Let conditions of Theorem* 1 *be satisfied and H.r.v.* ξ *is infinitely divisible with the ch.f.* (10). *In order that the series* (1) *converge it is necessary and sufficient that the series*

$$\sum_{n=N+1}^{\infty} nM\left(a_{n-1}^{-1} < \|y\| \leqslant a_n^{-1}\right)$$

converge.

Proof. If the series (1) is convergent then its sum also will be infinitely divisible with ch.f.

$$g(y) = \exp\left\{i(x_1, y) - \frac{1}{2}(S_1 y, y) + \int \left[e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|^2}\right] dM_1(x)\right\},$$

where $x_1 \in H$, $S_1 = \sum_{n=1}^{\infty} A_n^* S A_n$,

$$M_1(B) = \sum_{n=1}^{\infty} M(A_n^{*-1} B) \tag{11}$$

for any Borel set B which does not include some neighborhood of the origin (here A_n^* are conjugate operators to linear operators A_n, A_n^{*-1} are inverse operators to linear operators A_n^*). In a special case there must be $M_1(\|x\| > 1) < \infty$ and we have for $\varepsilon > 0$

$$\begin{aligned}\int\limits_{\|x\|>\varepsilon} dM_1(x) &= \sum_{n=1}^{\infty} \int\limits_{\|y\|>\varepsilon a_n^{-1}} dM(y) \\ &= \sum_{n=1}^{N} \int\limits_{\|y\|>\varepsilon a_n^{-1}} dM(y) + \sum_{n=N+1}^{\infty} \sum_{j=n}^{\infty} \int\limits_{\varepsilon a_{j-1}^{-1} < \|y\| \leqslant \varepsilon a_n^{-1}} dM(y) \\ &= C + \sum_{j=N+1}^{\infty} j M(\varepsilon a_{j-1}^{-1} < \|y\| \leqslant \varepsilon a_j^{-1}),\end{aligned} \tag{12}$$

here C is constant. By virtue of the equation (11) we have the *necessity*.

Sufficiency. By equations (11) and (12) we have that M_1 is a σ-finite measure with finite mass outside every neighborhood of the origin. So it is sufficient to show the convergence of the integral

$$I = \int\limits_{\|x\|<1} \|x\|^2 \, dM_1(x).$$

This follows from the next inequalities:

$$\begin{aligned}I &\leqslant \sum_{n=1}^{\infty} \int\limits_{\|A_n^* x\| \leqslant 1} \|A_n^* x\|^2 \, dM(x) \\ &= \sum_{n=1}^{N} a_n^2 \int\limits_{\|x\| \leqslant a_n^{-1}} \|x\|^2 \, dM(x) + \sum_{n=N+1}^{\infty} a_n^2 \int\limits_{\|x\| \leqslant a_N^{-1}} \|x\|^2 \, dM(x)\end{aligned}$$

$$+ \sum_{n=N+1}^{\infty} a_n^2 \sum_{j=N+1}^{n} \int_{a_{j-1}^{-1} < \|y\| \leqslant a_j^{-1}} \|y\|^2 \, dM(y)$$

$$= C_1 + \sum_{j=N+1}^{\infty} \sum_{n=j}^{\infty} a_n^2 \int_{a_{j-1}^{-1} < \|y\| \leqslant a_j^{-1}} \|y\|^2 \, dM(y)$$

$$\leqslant C_1 + C_2 \sum_{j=N+1}^{\infty} nM(a_{j-1}^{-1} < \|y\| \leqslant a_j^{-1}),$$

here C_1 and C_2 are constants.

Theorem is proved.

V. M. Kruglov has proved this nice result ([2]):

THEOREM. *Let there exists a constant* $C = C(\varphi)$ *such, that for all* $x, y \in H$ *inequality*

$$\varphi(x + y) \leqslant C\varphi(x)\varphi(y)$$

is true. The integral

$$\int_H \varphi(x) \, d\nu(x)$$

is finite if and only if the integral

$$\int_{\|x\| > \gamma} \varphi(x) \, dM(x)$$

is finite for some $\gamma > 0$.

From the Corollary 1 and from the Theorem 2 by using Kruglov's result or by using Theorem 3 we can conclude the next corollary's.

COROLLARY 2. *Let conditions of Corollary* 1 *be satisfied and H.r.v.* ξ *is infinitely divisible with the ch.f.* (10). *Then the series* (1) *converge if and only if*

$$\int_{\|x\| > 1} \|x\|^{\frac{1}{\alpha}} \, dM(x) < \infty.$$

COROLLARY 3. *Let conditions of Theorem* 2 *be satisfied and probability measure* ν *is the distribution of an infinitely divisible H.r.v.* ξ *with the ch.f.* (10). *The probability measure* ν *can be a component in A-decomposition* (7) *of some probability measure* μ *if and only if*

$$\int_{\|x\| > 1} \log \|x\| \, dM(x) < \infty. \qquad (13)$$

It follows that under condition (13) for all a-operators A with $0 < a < 1$ there exists a probability measure μ such that equation (7) is true.

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Apie tikimybinių matų Hilberto erdvėse A-skaidomumą

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Įrodoma, jog tikimybinis matas ν yra kurio nors tikimybinio mato μ A-skaidinio (7) komponentė tada ir tik tada kai konverguoja integralas (8). Čia tiesinis operatorius A turi savybę: su visais x iš Hilberto erdvės $\|Ax\| = a\|x\|$ ir $0 < a < 1$.