

# On extension of Donsker–Prokhorov invariance principle

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**Abstract.** The extension of Donsker–Prokhorov invariance principle for two-dimensional parameter summation process in Hölder space is provided for the case of independent non-identically distributed random variables.

*Keywords:* Brownian sheet, functional central limit theorem, Hölder space, invariance principle, triangular array, summation process.

## 1. Introduction

There exist a lot of extensions of Donsker–Prokhorov invariance principle. They are concerned with using different functional spaces, using multiparameter processes, introducing dependence and other topics. In this article the case of independent non-identically distributed random variables will be discussed. For one dimensional parameter case main results are Araujo and Gine [1] for continuous function space and Račkauskas and Suquet [7] for Hölder function space. Both results use adaptive construction of summation process. In two dimensional parameter case Bickel and Wichura [2] extended Donsker–Prokhorov invariance principle for variables with special variance structure in space  $D_2$  of functions “continuous from above with limits from below”. The result presented will generalize the result of Račkauskas and Suquet for two dimensional parameter case, by using process construction of Bickel and Wichura.

## 2. Notation and results

In this paper vectors  $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ , are typeset in italic bold. As a vector space  $\mathbb{R}^2$ , is endowed with the norm

$$|\mathbf{t}| = \max(|t_1|, |t_2|), \quad \mathbf{t} = (t_1, t_2) \in \mathbb{R}^2.$$

We define the Hölder space  $H_\alpha^q([0, 1]^2)$  as the vector space of functions  $x: [0, 1]^2 \rightarrow \mathbb{R}$  such that

$$\|x\|_\alpha := |x(0)| + \omega_\alpha(x, 1) < \infty,$$

with

$$\omega_\alpha(x, \delta) := \sup_{0 < |\mathbf{t}-\mathbf{s}| \leq \delta} \frac{|x(\mathbf{t}) - x(\mathbf{s})|}{|\mathbf{t} - \mathbf{s}|^\alpha} \xrightarrow{\delta \rightarrow 0} 0.$$

Endowed with the norm  $\|\cdot\|_\alpha$ ,  $H_\alpha^0([0, 1]^2)$  is a separable Banach space, see [3] or [4].

Define triangular array with double index as

$$(X_{n,ij}, 1 \leq i \leq I_n, 1 \leq j \leq J_n), \quad n \in \mathbb{N}$$

where for each  $n$  random variables are independent. Assume that  $EX_{n,ij} = 0$  and that  $\sigma_{n,ij}^2 := EX_{n,ij}^2 = a_{n,i}b_{n,j}$ , with  $\sum_{i=1}^{I_n} a_{n,i} = 1$  and  $\sum_{j=1}^{J_n} b_{n,j} = 1$ . For  $1 \leq k \leq I_n$ ,  $1 \leq l \leq J_n$  let

$$S_n(k, l) = \sum_{i=1}^k \sum_{j=1}^l X_{n,ij}, \quad A_{n,k} = \sum_{i=1}^k a_{n,i} \quad B_{n,l} = \sum_{j=1}^l b_{n,j}$$

For  $1 \leq i \leq I_n$ ,  $1 \leq j \leq J_n$  let

$$R_{n,ij} := [A_{n,i-1}, A_{n,i}] \times [B_{n,j-1}, B_{n,j}]. \quad (1)$$

Due to definition of  $a_{n,i}$  and  $b_{n,j}$  we get that  $R_{n,ij} \cap R_{n,kl} = \emptyset$ , if  $(i, j) \neq (k, l)$ ,  $\cup_{1 \leq i \leq I_n} \cup_{1 \leq j \leq J_n} R_{n,ij} = [0, 1]^2$  and  $\sum_{1 \leq i \leq I_n} \sum_{1 \leq j \leq J_n} |R_{n,ij}| = 1$ , where  $|A|$  denotes the Lebesgue measure of the set  $A \in \mathbb{R}^2$ .

Now define the summation process

$$\xi_n(t_1, t_2) = \sum_{1 \leq i \leq I_n} \sum_{1 \leq j \leq J_n} |R_{n,ij}|^{-1} |R_{n,ij} \cap [0, t_1] \times [0, t_2]| X_{n,ij}. \quad (2)$$

A Wiener sheet  $W(t)$ ,  $t \in [0, 1]^2$  is a mean zero Gaussian process with covariance function  $EW(t)W(s) = \min(t_1, s_1) \min(t_2, s_2)$ .

Now we can state our main result.

**THEOREM 1.** For  $0 < \alpha < 1/2$ , set  $q > 1/(1/2 - \alpha)$ . If

$$\max_{1 \leq i \leq I_n} a_{n,i} \rightarrow 0 \quad \text{with} \quad \max_{1 \leq j \leq J_n} b_{n,j} \rightarrow 0 \quad (3)$$

and

$$\lim_{n \rightarrow \infty} \sum_{1 \leq i \leq I_n} \sum_{1 \leq j \leq J_n} (a_{n,i} b_{n,j})^{-q\alpha} E|X_{n,ij}|^q = 0. \quad (4)$$

Then

$$\xi_n \xrightarrow[n \rightarrow \infty]{H_\alpha^0([0, 1]^2)} W.$$

This theorem generalizes Theorem 2 from [7] in a sense that if we “lose one dimension” the theorems will be the same.

### 3. Background and tools

#### 3.1. Summation process

As in [5] define operators

$$\Delta_i^{(1)} S_n(k, l) = S_n(i, l) - S_n(i - 1, l),$$

$$\Delta_j^{(2)} S_n(k, l) = S_n(k, j) - S_n(k, j - 1).$$

Note that they commute, and for each  $1 \leq i \leq I_n$ ,  $1 \leq j \leq J_n$

$$X_{n,ij} = \Delta_i^{(1)} \Delta_j^{(2)} S_n(k, l).$$

Define

$$A_n(t_1) = \max\{k \geq 0: A_{n,k} < t_1\}, \quad B_n(t_2) = \max\{l \geq 0: B_{n,l} < t_2\}.$$

Fix  $t = (t_1, t_2)$  and denote by  $I = A_n(t_1)$  and  $J = B_n(t_2)$ . Similar to [6] and [5] we get that  $\xi_n(t)$  has such representation:

$$\begin{aligned} \xi_n(t_1, t_2) &= S_n(I, J) + \frac{t_1 - a_{n,I}}{a_{I+1}} \Delta_{I+1}^{(1)} S_n(I, J) + \frac{t_2 - b_{n,J}}{b_{J+1}} \Delta_{J+1}^{(2)} S_n(I, J) \\ &+ \frac{t_1 - a_{n,I}}{a_{I+1}} \frac{t_2 - b_{n,J}}{b_{J+1}} \Delta_{I+1}^{(1)} S_n(I, J) \Delta_{J+1}^{(2)} S_n(I, J). \end{aligned} \quad (5)$$

#### 3.2. Tightness criteria

Let

$$U_j = \{k2^{-j}; 0 \leq k \leq 2^j\}^2$$

and let  $V_0 = U_0$ ,  $V_j = U_j \setminus U_{j-1}$ .  $V_j$  is the set of dyadic points  $\mathbf{v} = (k_1 2^{-j}, k_2 2^{-j})$  in  $U_j$  with at least one  $k_i$  odd. For any continuous function  $x$  define

$$\lambda_{0,\mathbf{v}}(x) = x(\mathbf{v}), \quad \mathbf{v} \in V_0,$$

$$\lambda_{j,\mathbf{v}}(x) = x(\mathbf{v}) - \frac{1}{2}(x(\mathbf{v}^-) + x(\mathbf{v}^+)), \quad \mathbf{v} \in V_j, \quad j \geq 1,$$

where

$$v_i^- = \begin{cases} v_i - 2^{-j}, & \text{if } k_i \text{ is odd,} \\ v_i, & \text{if } k_i \text{ is even;} \end{cases} \quad v_i^+ = \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd,} \\ v_i, & \text{if } k_i \text{ is even,} \end{cases}$$

for  $i = 1, 2$ .

As a tightness criterion we will use following theorem, which is a special case of Theorem 6 in [5].

**THEOREM 2.** *The sequence  $(Z_n)_{n \geq 1}$  of random elements of Hölder space  $H_\alpha([0, 1]^2)$  is asymptotically tight if following conditions hold:*

- (i) For each dyadic  $\mathbf{t} \in [0, 1]^2$ , sequence  $Z_n(\mathbf{t})$  is asymptotically tight.  
 (ii) For each  $\varepsilon > 0$

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left( \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(Z_n)| > \varepsilon \right).$$

#### 4. Proof of Theorem 1

##### 4.1. Finite-dimensional distributions

Fix  $\mathbf{t} = (t_1, t_2)$  and denote by  $I = A_n(t_1)$  and  $J = B_n(t_2)$ . Define process

$$\zeta_n(\mathbf{t}) = S_n(I, J).$$

This process coincides with the process considered by Bickel and Wichura [2]. Now from (5) we see that

$$E(\xi_n(\mathbf{t}) - \zeta_n(\mathbf{t}))^2 \leq \sum_{j=1}^J a_{I+1} b_{n,j} + \sum_{i=1}^I a_i b_{J+1} + a_{I+1} b_{J+1} \rightarrow 0, \text{ as } n \rightarrow \infty \quad (6)$$

due to (3). Furthermore since

$$\sum_{i=1}^{I_n} \sum_{j=1}^{J_n} E X_{n,ij}^2 \mathbf{1}\{|X_{n,ij}| \geq \varepsilon\} \leq \frac{1}{\varepsilon^{q-2}} \sum_{i=1}^{I_n} \sum_{j=1}^{J_n} E |X_{n,ij}|^q$$

we get that condition (4) ensures that Lindeberg condition is satisfied, thus conditions of Theorem 5 from [2] are satisfied and  $\zeta_n(\mathbf{t})$  finite-dimensional distributions converge to Wiener sheet finite-dimensional distributions. Now (6) ensures the same for process  $\xi_n(\mathbf{t})$ .

##### 4.2. Tightness

We will use Theorem 2. The condition (i) follows from convergence of finite dimensional distributions. We now prove condition (ii). Due to definition of  $\lambda_{j, \mathbf{v}}(\xi_n)$  it is easy to check, that (ii) holds, provided that, for every  $\varepsilon > 0$

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left( \sup_{j \geq J} 2^{\alpha j} \max_{\substack{r \in D_j \\ 0 \leq l \leq 2^j}} |\xi_n(r, s_l) - \xi_n(r^-, s_l)| > \varepsilon \right) = 0, \quad (7)$$

where  $D_j = \{2(l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}$ ,  $r^- = r - 2^{-j}$ ,  $s_l = l2^{-j}$ .

Denote  $\mathbf{v} = (r, s_l)$ ,  $\mathbf{v}^- = (r^-, s_l)$ . Also denote  $I = A_n(r)$ ,  $I^- = A_n(r^-)$ ,  $J = B_n(s_l)$ . Consider following configurations.

Case 1.  $A_n(r) = A_n(r^-)$ . Then

$$\xi_n(r, s_l) - \xi_n(r^-, s_l) = \frac{r - r^-}{a_{I+1}} \Delta_{I+1}^{(1)} S_n(I, J) + \frac{r - r^-}{a_{I+1}} \frac{s_l - b_{n,J}}{b_{J+1}} \Delta_{I+1}^{(1)} \Delta_{J+1}^{(2)} S_n(I, J).$$

Since

$$\frac{r - r^-}{a_{I+1}} \leq \frac{2^{-j\alpha}}{a_{I+1}^\alpha},$$

we have

$$|\xi_n(r, s_l) - \xi_n(r^-, s_l)| \leq 3 \cdot 2^{-j\alpha} \max_{\substack{1 \leq i \leq I_n \\ 1 \leq j \leq J_n}} \left| \frac{\Delta_i^{(1)} S_n(i, j)}{a_{n,i}^\alpha} \right|.$$

*Case 2.*  $A_n(r) = A_n(r^-) + 1$ . The same inequality holds with right side multiplied by 2.

*Case 3.*  $A_n(r) > A_n(r^-) + 1$ . Put  $u = (A_n, I, s_l)$ ,  $u^- = (A_n, I^-, s_l)$ . Then

$$|\xi_n(v) - \xi_n(v^-)| \leq |\xi_n(v) - \xi_n(u)| + |\xi_n(u) - \xi_n(u^-)| + |\xi_n(u^-) - \xi_n(v^-)|.$$

We have

$$|\xi_n(u) - \xi_n(u^-)| = S_n(I, J) - S_n(I^-, J) + \frac{s_l - b_{n,J}}{b_{J+1}} \Delta_{J+1}^{(2)} (S_n(I, J) - S_n(I^-, J))$$

thus

$$|\xi_n(u) - \xi_n(u^-)| \leq 3 \max_{1 \leq j \leq J_n} \sum_{i=I^-+2}^I \Delta_i^{(1)} S_n(i, j).$$

Using these inequalities we get that (7) will hold if

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \Pi_1(J, n; \varepsilon) = 0, \quad (8)$$

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \Pi_2(J, n; \varepsilon) = 0, \quad (9)$$

where

$$\Pi_1(J, n; \varepsilon) = P \left( \max_{\substack{1 \leq i \leq I_n \\ 1 \leq j \leq J_n}} \left| \frac{\Delta_i^{(1)} S_n(i, j)}{a_{n,i}^\alpha} \right| > \varepsilon \right). \quad (10)$$

and

$$\Pi_2(J, n; \varepsilon) = P \left( \sup_{j \geq J} 2^{-j\alpha} \max_{r \in D_j} \max_{1 \leq l \leq J_n} \sum_{k=A_n(r^-)+2}^{A_n(r)} |\Delta_k^{(1)} S_n(k, l)| > \varepsilon \right) \quad (11)$$

*Proof of (8).* Using Markov and Doob inequalities for  $q > 1/(1/2 - \alpha)$  we get

$$\Pi_1 \leq \sum_{i=1}^{I_n} \varepsilon^{-q} a_{n,i}^{-q\alpha} E |\Delta_i^{(1)} S_n(i, J_n)|^q$$

$$\begin{aligned} &\leq c \sum_{i=1}^{I_n} \varepsilon^{-q} a_{n,i}^{-q\alpha} \left( \left( \sum_{j=1}^{J_n} a_{n,i} b_{n,j} \right)^{q/2} + \sum_{j=1}^{J_n} E|X_{n,ij}|^q \right) \\ &\leq c \varepsilon^{-q} \left( \sum_{i=1}^{I_n} a_{n,i}^{q(1/2-\alpha)} + \sum_{i=1}^{I_n} \sum_{j=1}^{J_n} (a_{n,i} b_{n,j})^{-q\alpha} E|X_{n,ij}|^q \right). \end{aligned}$$

Now (8) follows from (3) and (4).

*Proof of (9).* Using Doob and Rosenthal inequalities and the definition of  $A_n(r)$  we have

$$\begin{aligned} \Pi_2(J, n, \varepsilon) &\leq \sum_{j \geq J} \sum_{r \in D_j} \varepsilon^{-q} 2^{\alpha j q} E \left( \max_{1 \leq l \leq J_n} \sum_{k=A_n(r^-)+2}^{A_n(r)} |\Delta_k^{(1)} S_n(k, l)| \right)^q \\ &\leq \sum_{j \geq J} \sum_{r \in D_j} \varepsilon^{-q} 2^{\alpha j q} c \left( \left( \sum_{k=A_n(r^-)+2}^{A_n(r)} \sum_{l=1}^{J_n} a_{n,k} b_{n,kl} \right)^{q/2} + \sum_{k=A_n(r^-)+2}^{A_n(r)} \sum_{l=1}^{J_n} E|X_{n,kl}|^q \right) \\ &\leq \sum_{j \geq J} \varepsilon^{-q} 2^{j(1+\alpha q - q/2)} + \sum_{j \geq J} \sum_{r \in D_j} \varepsilon^{-q} 2^{\alpha j q} \sum_{k=A_n(r^-)+2}^{A_n(r)} \sum_{l=1}^{J_n} E|X_{n,kl}|^q. \end{aligned}$$

The first of the summands does not depend on  $n$  and tends to zero, when  $J$  tends to infinity, since  $q > 1/(1/2 - \alpha)$ . Denote by  $I(J, n, q)$  the second sum without the constant  $c\varepsilon^{-q}$ . By changing the order of summation we get

$$I(J, n, q) = \sum_{k=1}^{I_n} \sum_{l=1}^{J_n} E|X_{n,kl}|^q \sum_{j \geq J} 2^{\alpha j q} \sum_{r \in D_j} \mathbf{1}\{A_n(r^-) + 1 < k \leq A_n(r)\}.$$

The inner sum can be estimated identically to [7], with the slight change of notation.

$$\sum_{j \geq J} 2^{\alpha j q} \sum_{r \in D_j} \mathbf{1}\{A_n(r^-) + 1 < k \leq A_n(r)\} \leq \frac{2^{q\alpha}}{2^{q\alpha} - 1} a_k^{-\alpha q}. \quad (12)$$

Substituting (12) we get

$$I(J, n, q) \leq c \sum_{k=1}^{I_n} \sum_{l=1}^{J_n} a_k^{-q\alpha} E|X_{n,kl}|^q \leq \sum_{k=1}^{I_n} \sum_{l=1}^{J_n} (a_k b_l)^{-q\alpha} E|X_{n,kl}|^q,$$

and we see that (11) ensures (9), thus condition (ii) holds.

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## REZIUMĒ

***V. Zemlys. Apie Donsker–Prokhorov invariantiškumo principo apibendrinimą***

Šiame straipsnyje yra apibendrinamas Donsker–Prokhorov invariantiškumo principas dvimačio indekso sumavimo procesui serijų schemeje.