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Large deviations for endomorphisms of torus

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Let Ω_2 be a two-dimensional torus, $\vec{x}, \vec{x}_1, \vec{x}_2, \ldots \in \Omega_2$. The distance for elements of torus is defined by

$$\begin{aligned}\rho(\vec{x}_1, \vec{x}_2) &= \rho(C_{\vec{x}_1}, C_{\vec{x}_2})\\ &= \inf\Big\{\rho\big((x_{11}, x_{12}), (x_{21}, x_{22})\big)\colon (x_{11}, x_{12}) \in C_{\vec{x}_1},\ (x_{21}, x_{22}) \in C_{\vec{x}_2}\Big\},\end{aligned}$$

where $C_{\vec{x}_1}$ and $C_{\vec{x}_2}$ are equivalence classes in $\mathbb{R}^2$ modulo $\mathbb{Z}^2$.

Let $W = \|a_{ij}\|$ be a square matrix of integer elements. Let the endomorphism $T\colon \Omega_2 \to \Omega_2$ be defined by

$$T\vec{x} = \vec{x}\,W \pmod 1.$$

If the eigenvalues of the matrix W are not equal to 1, then the mapping T is invariant with respect to Lebesgue measure μ_2 on $\mathbb{R}^2$.

If det $W \neq \pm 1$, then the mapping T is an automorphism.

These and other properties of mappings of torus can be found in [1,2]. This article is a continuation of investigation by D. Moskvin [2] and of earlier papers by authors [3,4].

Let

$$\vec{\xi} = \vec{\xi}(t) = \big\{(\varphi(t), \psi(t)),\ a \leqslant t \leqslant b\big\} \tag{1}$$

be a smooth parametric curve on Ω_2, $\Phi(x)$ be a standard normal distribution function. Let $\vec{w}_i = (w_{i1}, w_{i2})$, $i = 1, 2$, be eigenvectors of W corresponding to the eigenvalues θ_1 and θ_2, $|\theta_1| > 1$, $|\theta_2| = |\theta_1|^{-1}$.

Let $h(\vec{x})$, $\vec{x} \in \Omega_2$, be a real function of two arguments satisfying the condition

$$\big|h(\vec{x}_1) - h(\vec{x}_2)\big| \leqslant H\rho(\vec{x}_1, \vec{x}_2), \tag{2}$$

where $H \geqslant \{\max|h(\vec{x})|,\ \vec{x} \in \Omega_2\}$,

$$\int_{\Omega_2} h(\vec{x})\,\mathrm{d}\vec{x} = 0. \tag{3}$$

Let us consider

$$S_n(\vec{x}) = \sum_{k=0}^{n-1} h(\vec{x}\, W^k), \quad Z_n(\vec{x}) = \frac{1}{\sigma\sqrt{n}} S_n(\vec{x}),$$

where

$$\sigma^2 = \lim_{n\to\infty} \int_{\Omega_2} \Big(\frac{1}{\sqrt{n}} S_n(\vec{x})\Big)^2 \mathrm{d}\vec{x}, \quad \sigma^2 > 0.$$

Two distribution functions can be defined:

$$F_n(x) = \mu_2\big(\vec{x} \in \Omega_2\colon Z_n(\vec{x}) < x\big),$$

$$F_{n,\xi}(x) = \frac{1}{b-a}\mu_1\big(t \in [a,b]\colon Z_n(\vec{\xi}\,) < x\big).$$

The main result of this article is

THEOREM 1. *Let the function $h(\vec{x})$, $\vec{x} \in \Omega_2$, and the curve $\vec{\xi}(t)$, $t \in [a,b]$, satisfy the above mentioned conditions. Then in the interval*

$$0 \leqslant x \leqslant \frac{c\sqrt{n}}{\ln^2 n}, \quad c > 0,$$

the following relations for the large deviations are valid:

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\{L(x)\}\Big(1 + \mathrm{O}\Big(\frac{x \ln^2 n}{\sqrt{n}}\Big)\Big),$$

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\{L(-x)\}\Big(1 + \mathrm{O}\Big(\frac{x \ln^2 n}{\sqrt{n}}\Big)\Big),$$

where

$$L(x) = \sum_{k=3}^{\infty} \lambda_k x^k,$$

the coefficients λ_k are expressed in terms of cumulants of the sum Z_n, and coincide with the coefficients of the classical Cramér–Petrov series.

The analogous results are valid for the distribution function $F_{n,\xi}(x)$.

The following auxiliary propositions are needed for the proof of Theorem 1.

LEMMA 1. *Let the functions $f(\vec{x})$ and $g(\vec{x})$, $\vec{x} \in \Omega_2$, satisfy the condition* (2) *with the constants A and B respectively,*

$$\max_{\vec{x}\in\Omega_2} |f(\vec{x})| \leqslant A, \quad \max_{\vec{x}\in\Omega_2} |g(\vec{x})| \leqslant B.$$

Then

$$\int_a^b f(\vec{\xi})g(\vec{\xi}\,W^m)\,\mathrm{d}t = \int_a^b f(\vec{\xi})\,\mathrm{d}t \cdot \int_{\Omega_2} g(\vec{x})\,\mathrm{d}\vec{x} + \mathrm{O}\Big(\frac{mAB}{\varepsilon^3\theta_1^m}\Big), \tag{4}$$

$$\int_{\Omega_2} f(\vec{x})g(\vec{x}\,W^m)\,\mathrm{d}\vec{x} = \int_{\Omega_2} f(\vec{x})\,\mathrm{d}\vec{x} \cdot \int_{\Omega_2} g(\vec{x})\,\mathrm{d}\vec{x} + \mathrm{O}\Big(\frac{mAB}{\theta_1^m}\Big), \tag{5}$$

where $\vec{\xi}(t)$ *is a curve defined by* (1), θ_1 *is an eigenvalue,* $\theta_1 > \theta_2$,

$$\varepsilon = \min_{t\in[a,b]} \big|w_{21}\varphi'(t) - w_{22}\psi'(t)\big| > 0.$$

Proof. See [2].

Denote

$$S_{k,l} = S_{k,l}(\vec{x}) = \sum_{k\leqslant i\leqslant l} h(\vec{x}\,W^i).$$

Let q and m be natural numbers which will be chosen later, $p = [\frac{n}{q} + m]^{-1}$.
Denote

$$\eta_k(\vec{x}) = S_{(k-1)(q+m)+1,kq+(k-1)m}, \quad 1 \leqslant k \leqslant p,$$
$$\eta_k^0(\vec{x}) = S_{kq+(k-1)m+1,k(q+m)}, \quad 1 \leqslant k \leqslant p,$$
$$\eta_{p+1}^0(\vec{x}) = S_{p(q+m)+1,n}.$$

Then the sum $S_n(\vec{x})$ can be expressed as follows:

$$S_n(\vec{x}) = \sum_{k=1}^{p} \eta_k(\vec{x}) + \sum_{k=1}^{p+1} \eta_k^0(\vec{x}) = \zeta_n(\vec{x}) + \zeta_n^0(\vec{x}).$$

LEMMA 2. *The characteristic function*

$$f_n(t) = \int_{\Omega_2} \exp\big(it\zeta_n(\vec{x})\big)\,\mathrm{d}\vec{x}$$

can be evaluated as follows:

$$f_n(t) = \Big(\int_{\Omega_2} \exp\big(it\eta_1(\vec{x})\,\mathrm{d}\vec{x}\big)\Big)^p + \mathrm{O}\Big(\frac{n^2tH}{\theta_1^m}\Big).$$

Proof. See [2].

LEMMA 3. *The following estimate is valid:*

$$\mathbf{D}S_n(\vec{x}) = \sigma^2 n + \mathrm{O}(1). \tag{6}$$

Proof. By making use of (3) and regrouping summands we get:

$$\begin{aligned}
\mathbf{D}S_n(\vec{x}) &= \mathbf{E}S_n^2(\vec{x}) \\
&= n\int_{\Omega_2} h^2(\vec{x})\,\mathrm{d}\vec{x} + 2n\sum_{j=1}^{n-1}\Big(1-\frac{j}{n}\Big)\int_{\Omega_2} h(\vec{x})h\big(\vec{x}\,W^j\big)\,\mathrm{d}\vec{x} \\
&= n\bigg(\int_{\Omega_2} h^2(\vec{x})\,\mathrm{d}\vec{x} + 2\sum_{j=1}^{\infty}\int_{\Omega_2} h(\vec{x})h\big(\vec{x}\,W^j\big)\,\mathrm{d}\vec{x}\bigg) \\
&\quad - 2n\sum_{j=n}^{\infty}\int_{\Omega_2} h(\vec{x})h\big(\vec{x}\,W^j\big)\,\mathrm{d}\vec{x} - 2\sum_{j=1}^{n-1} j\int_{\Omega_2} h(\vec{x})h\big(\vec{x}\,W^j\big)\,\mathrm{d}\vec{x}.
\end{aligned}$$

Having in mind the definition of σ^2 and taking $f(\vec{x}) = g(\vec{x}) = h(\vec{x})$ in (5) (Lemma 1) we get the estimate (6).

Let $\Gamma_k(S_n)$ denote the cumulant of k-th order of the sum $S_n(\vec{x})$. The estimate $\Gamma_k(S_n) = \mathrm{O}(n)$ is known [2]. We will show now the dependence of this estimate on the properties of $h(\vec{x})$ and on the order of the cumulant.

LEMMA 4. *The following estimate is valid:*

$$\Gamma_k(S_n) \leqslant H_0 H^k k!\big(\ln^2 n\big)^{k-2} n, \qquad H_0 > 0.$$

Proof. Let $\hat{\mathbf{E}}X_{t_1}\ldots X_{t_k}$ be the centered moment of the k-th order. The estimates of such moments are very important in limit theorems for sums of dependent random variables. Analogous to Theorem 4.4 in [5], based on (4) and Lemma 2, we can prove that

$$\big|\hat{\mathbf{E}}\xi_{t_1}\ldots\xi_{t_k}\big| \leqslant 2^{k-1}H^k\prod_{j=1}^{r-1}\frac{l_{j+1}-l_j}{\theta_1^{l_{j+1}-l_j}}. \tag{7}$$

Making use of the definition of cumulants, Lemma 1.1 in [5], and the expression of cumulants in terms of centered moments we get:

$$\Gamma_k(S_n) = \sum_{1\leqslant t_1,\ldots,t_k\leqslant n}\Gamma(X_{t_1},\ldots,X_{t_k}),$$

$$\Gamma(X_{t_1},\ldots,X_{t_k}) = \sum_{\nu=1}^{k}(-1)^{\nu-1}\sum_{\cup_{p=1}^{\nu}I_p=I}N_\nu(I_1,\ldots,I_\nu)\prod_{p=1}^{\nu}\hat{\mathbf{E}}(X_{I_p}).$$

Here the integers $N_\nu(I_1,\ldots,I_\nu)$,

$$0 \leqslant N_\nu(I_1,\ldots,I_\nu) \leqslant (\nu-1)!,$$

depend on ν-block partition $\{I_1, \ldots, I_\nu\}$ of the set $I = \{t_1, \ldots, t_k\}$ only, and

$$\hat{\mathbf{E}}(X_{I_p}) = \hat{\mathbf{E}}(X_{i_1} \ldots X_{i_p}).$$

Analogous to Theorem 4.11 in [5] we get from the inequality (7):

$$\left|\Gamma(\xi_{t_1}, \ldots, \xi_{t_k})\right| \leqslant (k-1)! \cdot 2^{k-1} H^k \prod_{j=1}^{r-1} \frac{l_{j+1} - l_j}{\theta_1^{l_{j+1}-l_j}}.$$

Now we take $q = [\omega_1 \ln n]$, $m = [\omega_2 \ln n]$ ($\omega_1 > 0$, $\omega_2 > 0$). Making use of the above listed estimates, in a way analogous to Theorem 4.1.9 in [5], we get the proposition of Lemma 4.

From the estimates of cumulants of the sum $S_n(\vec{x})$ we get the estimates of cumulants of the normed sum $Z_n(\vec{x})$:

$$\Gamma_k(Z_n) \leqslant H_0^* H^k k! \left(\frac{\ln n}{\sqrt{n}}\right)^{k-2}.$$

Making use of Lemmas 1–4 and Lemma 2.3 in [5] we obtain the proof of our main Theorem 1.

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REZIUMĖ

B. Kryžienė, G. Misevičius. Dvimačio toro endomorfizmų didieji nuokrypiai

Darbe suformuluotos keturios lemos ir jų pagalba įrodyta dvimačio toro transformacijų $\{\vec{x} W^k\}$, $k = 0, 1, 2, \ldots$, $\vec{x} \in \Omega_2$, didžiųjų nuokrypių ribinė teorema. Įrodymui panaudoti žinomi D. Moskvino ir V. Statulevičiaus centruotų momentų ir semiinvariantų įverčiai,